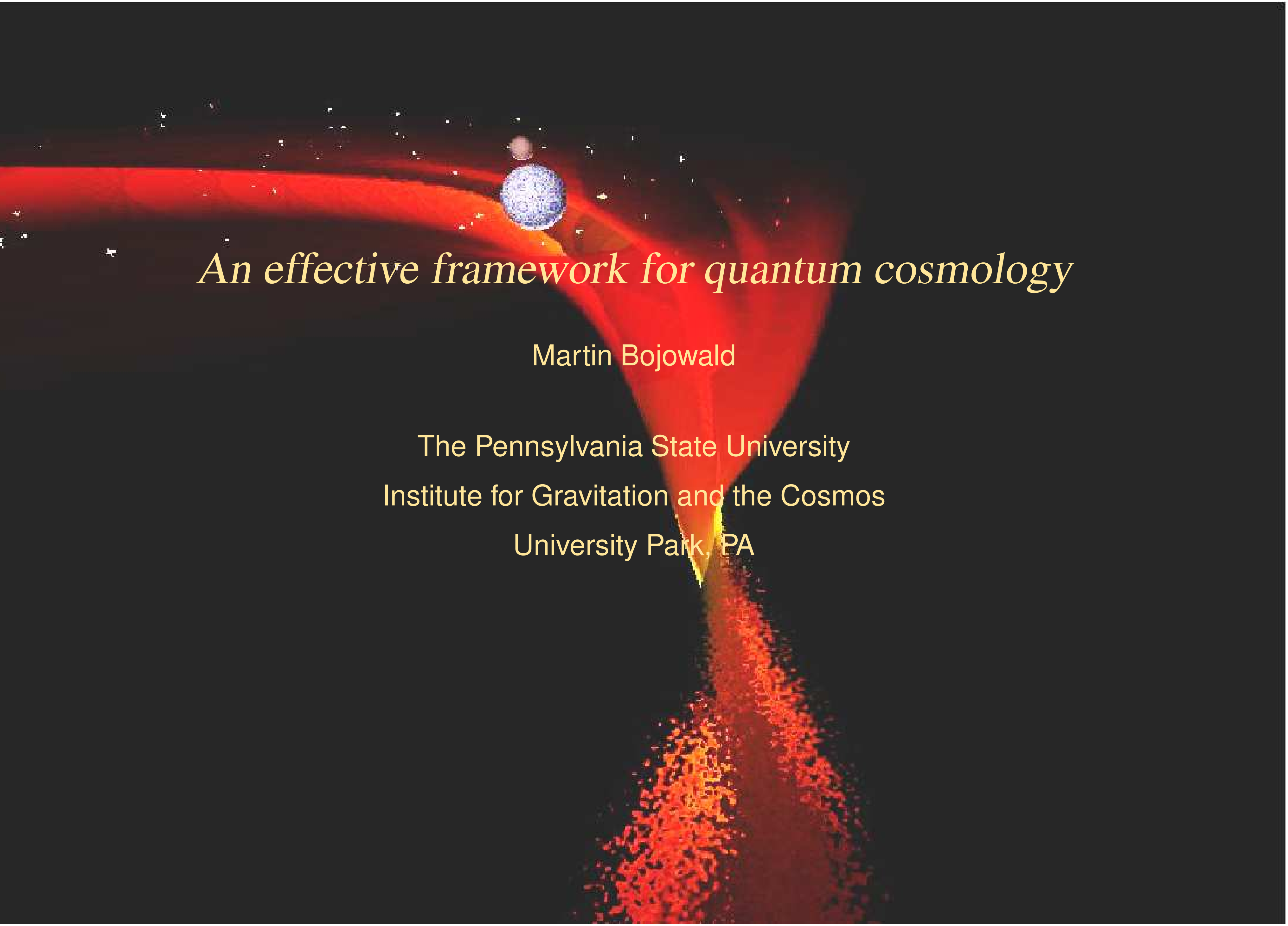

An effective framework for quantum cosmology

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Effective theoy

Powerful way to check whether physical predictions are meaningful and generic.

Not just “quantum corrections.” Effective equations contain information about interacting vacuum (or other relevant) states.

Quantum cosmology:

- → No clear ground state. Class of relevant states?
- → Symmetries important. For instance, higher-curvature effective action expected for quantum gravity.

  Assumes classical space-time structure.
  (Anomaly problem.)
- → Problem of time. Effective equations of motion?

References: arXiv:1209.3403, arXiv:1404.1018

Canonical effective theory: Starting point

Ehrenfest equations

$$\frac{\mathrm{d}\langle\hat{x}\rangle}{\mathrm{d}t} = \frac{\langle\hat{p}\rangle}{m}$$

$$\frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t} = -\langle V'(\hat{x})\rangle = -V'(\langle\hat{x}\rangle) - \frac{1}{2}V'''(\langle\hat{x}\rangle)(\Delta x)^2 + \cdots$$

Fluctuation dynamical: $\mathrm{d}(\Delta x)^2/\mathrm{d}t = 2C_{xp}/m$ proportional to covariance.

Infinitely many coupled ordinary differential equations for expectation values and moments

$$\langle(\hat{x} - \langle\hat{x}\rangle)^a(\hat{p} - \langle\hat{p}\rangle)^b\rangle_{\mathrm{symm}}$$

Need to know some dynamical state properties to compute quantum corrections $-\frac{1}{2}V'''(\langle\hat{x}\rangle)(\Delta x)^2 + \cdots$.

Effective phase space

Parameterize state by expectation values $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ of basic operators and moments

$$G^{a,n} = \langle(\hat{q}-\langle\hat{q}\rangle)^a(\hat{p}-\langle\hat{p}\rangle)^{n-a}\rangle_{\mathrm{symm}}$$

Commutator of operators determines Poisson bracket of moments.

$$\{\langle\hat{A}\rangle,\langle\hat{B}\rangle\} = \frac{\langle[\hat{A},\hat{B}]\rangle}{i\hbar}$$

Hamiltonian evolution for $\langle\hat{q}\rangle$, $\langle\hat{p}\rangle$ and $G^{a,n}$ generated by

$$\begin{aligned}\langle\hat{H}\rangle(\langle\hat{q}\rangle,\langle\hat{p}\rangle,G^{a,n}) &= \langle H(\langle\hat{q}\rangle+(\hat{q}-\langle\hat{q}\rangle),\langle\hat{p}\rangle+(\hat{p}-\langle\hat{p}\rangle))\rangle \\ &= H(\langle\hat{q}\rangle,\langle\hat{p}\rangle)+\sum_{n=2}^{\infty}\sum_{a=0}^{n}\frac{1}{a!b!}\frac{\partial^n H(\langle\hat{q}\rangle,\langle\hat{p}\rangle)}{\partial\langle\hat{q}\rangle^a\partial\langle\hat{p}\rangle^{n-a}}G^{a,n}\end{aligned}$$

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Equations of motion

Anharmonic oscillator $V(q) = \frac{1}{2}m\omega^2 q^2 + U(q)$:

$$
\begin{aligned}
\dot{q} &= \frac{p}{m} \\
\dot{p} &= -m\omega^2 q - U'(q) - \sum_n \frac{1}{n!}\left(\frac{\hbar}{m\omega}\right)^{n/2} U^{(n+1)}(q)\tilde{G}^{0,n} \\
\dot{\tilde{G}}^{a,n} &= -a\omega\tilde{G}^{a-1,n} + (n-a)\omega\tilde{G}^{a+1,n} - a\frac{U''(q)}{m\omega}\tilde{G}^{a-1,n} \\
&\quad + \frac{\sqrt{\hbar}aU'''(q)}{2(m\omega)^{\frac{3}{2}}}\tilde{G}^{a-1,n-1}\tilde{G}^{0,2} + \frac{\hbar aU''''(q)}{3!(m\omega)^2}\tilde{G}^{a-1,n-1}\tilde{G}^{0,3} \\
&\quad - \frac{a}{2}\left(\frac{\sqrt{\hbar}U'''(q)}{(m\omega)^{\frac{3}{2}}}\tilde{G}^{a-1,n+1} + \frac{\hbar U''''(q)}{3(m\omega)^2}\tilde{G}^{a-1,n+2}\right) + \cdots
\end{aligned}
$$

∞ly many coupled equations for ∞ly many variables.

Low-energy effective action

To second adiabatic order, as second order equation of motion:

$$\left(m+\frac{\hbar U'''(q)^2}{32m^2\omega^5\left(1+\frac{U''(q)}{m\omega^2}\right)^{\frac{5}{2}}}\right)\ddot{q}$$
$$+\frac{\hbar\dot{q}^2\left(4m\omega^2U'''(q)U''''(q)\left(1+\frac{U''(q)}{m\omega^2}\right)-5U'''(q)^3\right)}{128m^3\omega^7\left(1+\frac{U''(q)}{m\omega^2}\right)^{\frac{7}{2}}}$$
$$+m\omega^2q+U'(q)+\frac{\hbar U'''(q)}{4m\omega\left(1+\frac{U''(q)}{m\omega^2}\right)^{\frac{1}{2}}}=0\,.$$

as it results from

$$\Gamma_{\rm eff}[q(t)] = \int {\rm d}t\left(\frac{1}{2}\left(m+\frac{\hbar U'''(q)^2}{2^5m^2\left(\omega^2+m^{-1}U''(q)\right)^{\frac{5}{2}}}\right)\dot{q}^2 -\frac{1}{2}m\omega^2q^2-U(q)-\frac{\hbar\omega}{2}\left(1+\frac{U''(q)}{m\omega^2}\right)^{\frac{1}{2}}\right)$$

Higher time derivatives

Higher adiabatic order: [with S Brahma, E Nelson: arXiv:1208.1242]

$$\begin{aligned}\ddot{q} &= -\omega^2 q - U'(q)/m \\ &\quad -\frac{\hbar}{2m^2\omega}U'''(q)\left(f(q,\dot{q}) + f_1(q,\dot{q})\ddot{q} + f_2(q)\ddot{q}^2 + f_3(q,\dot{q})\dddot{q} + f_4(q)\ddddot{q}\right) \\ &\quad + \cdots\end{aligned}$$

where

$$\begin{aligned}f &= \frac{1}{2}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-1/2} + \frac{U''''(q)\dot{q}^2}{16m\omega^4}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-5/2} \\ &\quad -\frac{5(U'''(q))^2\dot{q}^2}{64m^2\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} - \frac{U''''''(q)\dot{q}^4}{64m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} \\ &\quad +\frac{21(U''''(q))^2\dot{q}^4}{256m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} + \frac{7U'''''(q)U'''(q)\dot{q}^4}{64m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} \\ &\quad -\frac{231U''''(q)(U'''(q))^2\dot{q}^4}{512m^3\omega^{10}}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-11/2} + \frac{1155(U'''(q))^4\dot{q}^4}{4096m^4\omega^{12}}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-13/2}\end{aligned}$$

Effective theories and states

- → Low-energy effective action: Local.

  Adiabatic expansion assumes slowly-varying $G^{a,n}$.
  Analogous to derivative expansion in time.

  Can solve $\dot{G}^{a,n} = \cdots$ for $G^{a,n}(\langle\hat{q}\rangle, \langle\hat{p}\rangle)$ and insert in $\langle\dot{\hat{p}}\rangle = \cdots$

- → Initial values used for $\dot{G}^{a,n}$ equations specify state used to expand around. Sensitive to class of states considered.

  Adiabaticity near harmonic or free vacuum,
  but may not be consistent for more general systems.

- → Expansion in $\hbar$ more general. Semiclassical: $G^{a,n} \sim O(\hbar^{n/2})$.

  Gives finitely many equations to each order: Independent quantum degrees of freedom $G^{a,n}$ coupled to $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$.

  "Auxiliary" degrees of freedom for non-local effective action.

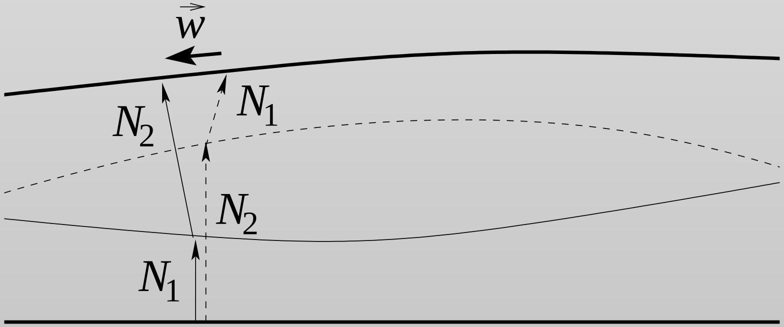
Effective canonical quantum gravity

Higher time derivatives expected from curvature invariants.

Minisuperspace models of quantum cosmology:
One time translation generator, Hamiltonian *constraint.*

Canonical quantum gravity: Operators for Hamiltonian and diffeomorphism constraint with *first-class off-shell algebra.*

Quantum version of hypersurface deformations in space-time:



$$\begin{aligned}
\{D[w_1^a], D[w_2^a]\} &= -D[\mathcal{L}_{w_2} w_1^a] \\
\{H[N], D[w^a]\} &= -H[w^a \nabla_a N] \\
\{H[N_1], H[N_2]\} &= D[h^{ab}(N_1 \nabla_b N_2 - N_2 \nabla_b N_1)]
\end{aligned}$$

Background versus quantum gravity

Cosmological (and other) phenomenology:

- → Metric and matter perturbations on a background. *General covariance* implies gauge transformations of modes.
- → May fix the gauge. *Field theory on a background.*

Gauge transformations and dynamics generated by the same constraints D and H.

- → Quantum gravity: Dynamics *and* gauge transformations for modes, potentially quantum corrected.
- → Quantum field theory on a background: Quantize only the dynamics of modes, after gauge fixing or other method to eliminate *classical* coordinate freedom.

May lead to different outcomes.

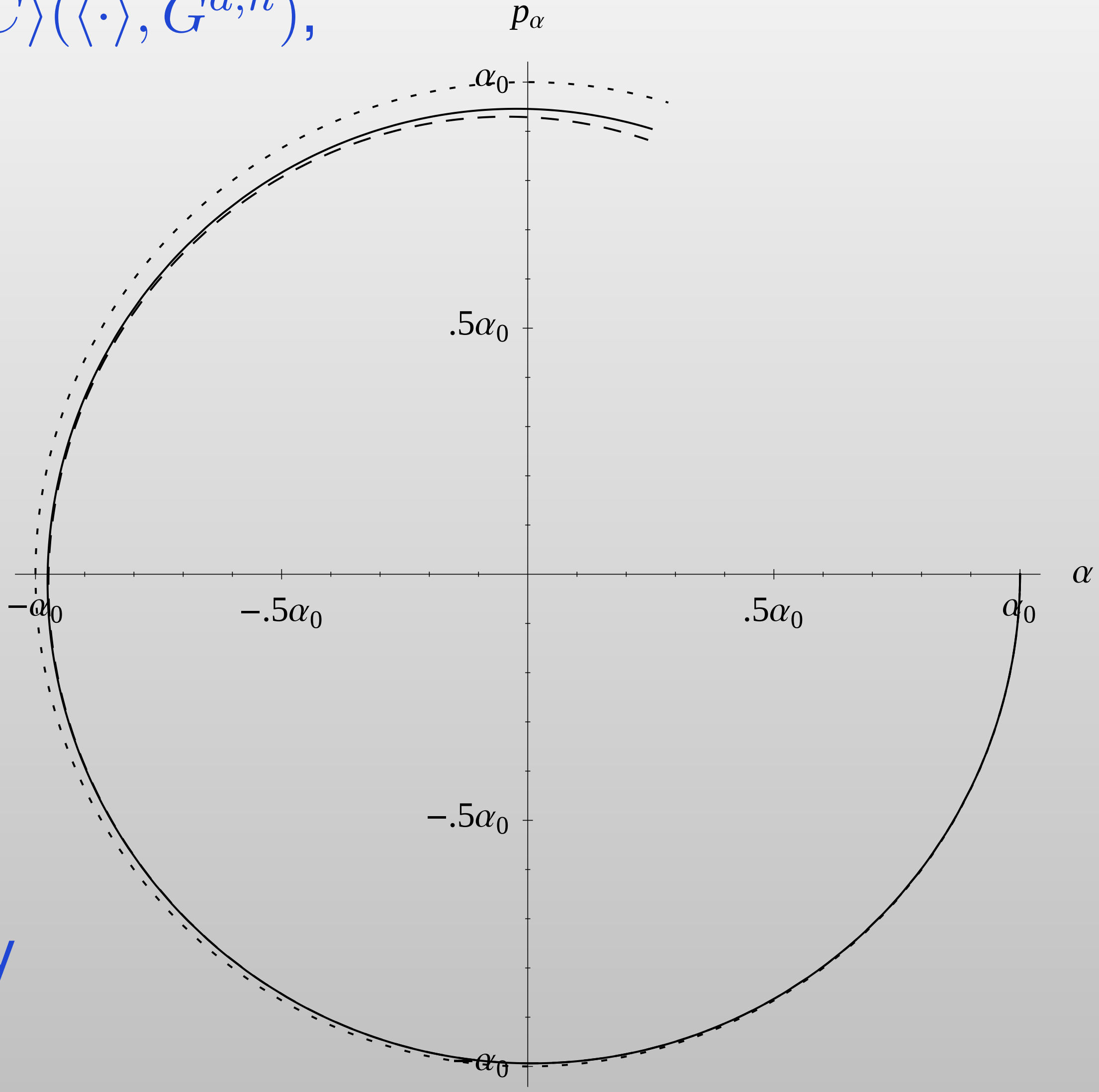
Effective constraints

Dynamics by effective Hamiltonian $\langle\hat{H}\rangle(\langle\cdot\rangle, G^{a,n})$
or effective constraints $\langle\widehat{\mathrm{pol}}\hat{C}\rangle(\langle\cdot\rangle, G^{a,n})$,
$\widehat{\mathrm{pol}}$ polynomial in $\hat{O} - \langle\hat{O}\rangle$
for all basic operators $\hat{O}$.

$\longrightarrow$ Moments of physical states:
Implement constraints, require real observables.

$\longrightarrow$ Local time:
Change internal time by gauge transformation.

$\longrightarrow$ Extension to field theory in progress.



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Anomaly problem

[with S. Brahma: arXiv:1407.4444]

Systems with several classical constraints, $[\hat{C}_I, \hat{C}_J] = \hat{f}_{IJ}^K \hat{C}_K$:
Effective constraints $C_{I,\mathrm{pol}} = \langle \widehat{\mathrm{pol}} \hat{C}_I \rangle \approx 0$.

$\{\langle \widehat{\mathrm{pol}}_A \hat{C}_I \rangle, \langle \widehat{\mathrm{pol}}_B \hat{C}_J \rangle\} = -i\hbar^{-1} \langle [\widehat{\mathrm{pol}}_A \hat{C}_I, \widehat{\mathrm{pol}}_B \hat{C}_J] \rangle$ first class if $[\hat{C}_I, \hat{C}_J]$ first class.

Classical structure functions $f_{IJ}^K(x_i)$ fully determine structure functions of $C_{I,1}$ with $\widehat{\mathrm{pol}} = 1$:

$$\{C_{I,1}, C_{J,1}\} = \langle \hat{f}_{IJ}^K \hat{C}_K \rangle = f_{IJ}^K(\langle \hat{x}_i \rangle) C_{K,1} + \sum_j \frac{\partial f_{IJ}^K(\langle \hat{x}_i \rangle)}{\partial \langle \hat{x}_j \rangle} C_{K,x_j} + \cdots$$

(Expand $\langle \hat{f}_{IJ}^K \hat{C}_K \rangle = \langle f_{IJ}^K(\langle \hat{x}_i \rangle + (\hat{x}_i - \langle \hat{x}_i \rangle)) \hat{C}_K \rangle$ with $f_{IJ}^K(\langle \hat{x}_i \rangle + (\hat{x}_i - \langle \hat{x}_i \rangle)) = f_{IJ}^K(\langle \hat{x}_i \rangle) + \cdots$.)

Hypersurface deformations

→ Effective algebra

$$\{H[N_1], H[N_2]\} = D[h^{ab}(N_1\nabla_b N_2 - N_2\nabla_b N_1)] + \cdots$$

if classical constraint algebra represented without modifications ($\hat{f}_{IJ}^K = f_{IJ}^K(\hat{x}_i)$).

Suggests higher-curvature effective actions.

→ Difficult to obtain first-class $[\hat{C}_I, \hat{C}_J]$ for gravity.

Some results in loop quantum gravity,
but with modifications/regularizations: $\hat{f}_{IJ}^K \neq f_{IJ}^K(\hat{x}_i)$.

Modified constraint algebra.
Modified hypersurface deformations.

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Holonomy modification

Higher-curvature corrections

$$\frac{8\pi G}{3}\rho = \mathcal{H}^2(1+O(\ell^2\mathcal{H}^2)) + O(\ell^2\dot{\mathcal{H}}^2) + \cdots$$

Loop quantum cosmology:

$$\frac{8\pi G}{3}\rho = \frac{\sin(\ell\bar{\mathcal{H}})^2}{\ell^2} = \mathcal{H}^2\left(1+O(\ell^2\mathcal{H}^2)\right)$$

Effective Friedmann equation ($\rho_{\rm QG} = 3/(8\pi G\ell^2)$)

$$\mathcal{H}^2 = \frac{8\pi G}{3}\rho\left(1-\frac{\rho}{\rho_{\rm QG}}\right) + \cdots$$

Part of covariant theory?

Holonomies and space-time

[Reyes; Barrau, Cailleteau, Grain, Mielczarek]

$\mathcal{H}^2 \longrightarrow f(\mathcal{H})$ anomaly-free if constraint algebra deformed to

$$\{H[N_1], H[N_2]\} = D[\beta h^{ab}(N_1\nabla_b N_2 - N_2\nabla_b N_1)]$$

with

$$\beta(\mathcal{H}) = \frac{1}{2}\frac{\mathrm{d}^2 f(\mathcal{H})}{\mathrm{d}\mathcal{H}^2}$$

Example: $\beta(\mathcal{H}) = \cos(2\ell\mathcal{H})$ for $f(\mathcal{H}) = \ell^{-2}\sin^2(\ell\mathcal{H})$.

→ Well-defined and consistent canonical effective theory.

But no classical or Riemannian space-time
(unless field redefinition).

→ Signature change: $\beta(\mathcal{H}) < 0$ around maximum of $f(\mathcal{H})$.

Space-time structure

Poisson bracket

$$\{H[N_1], H[N_2]\} = D[\beta h^{ab}(N_1 \nabla_b N_2 - N_2 \nabla_b N_1)]$$

with $\beta \neq 1$ implies that $H[N]$ generates gauge transformations of h_{ab} different from standard coordinate transformations.

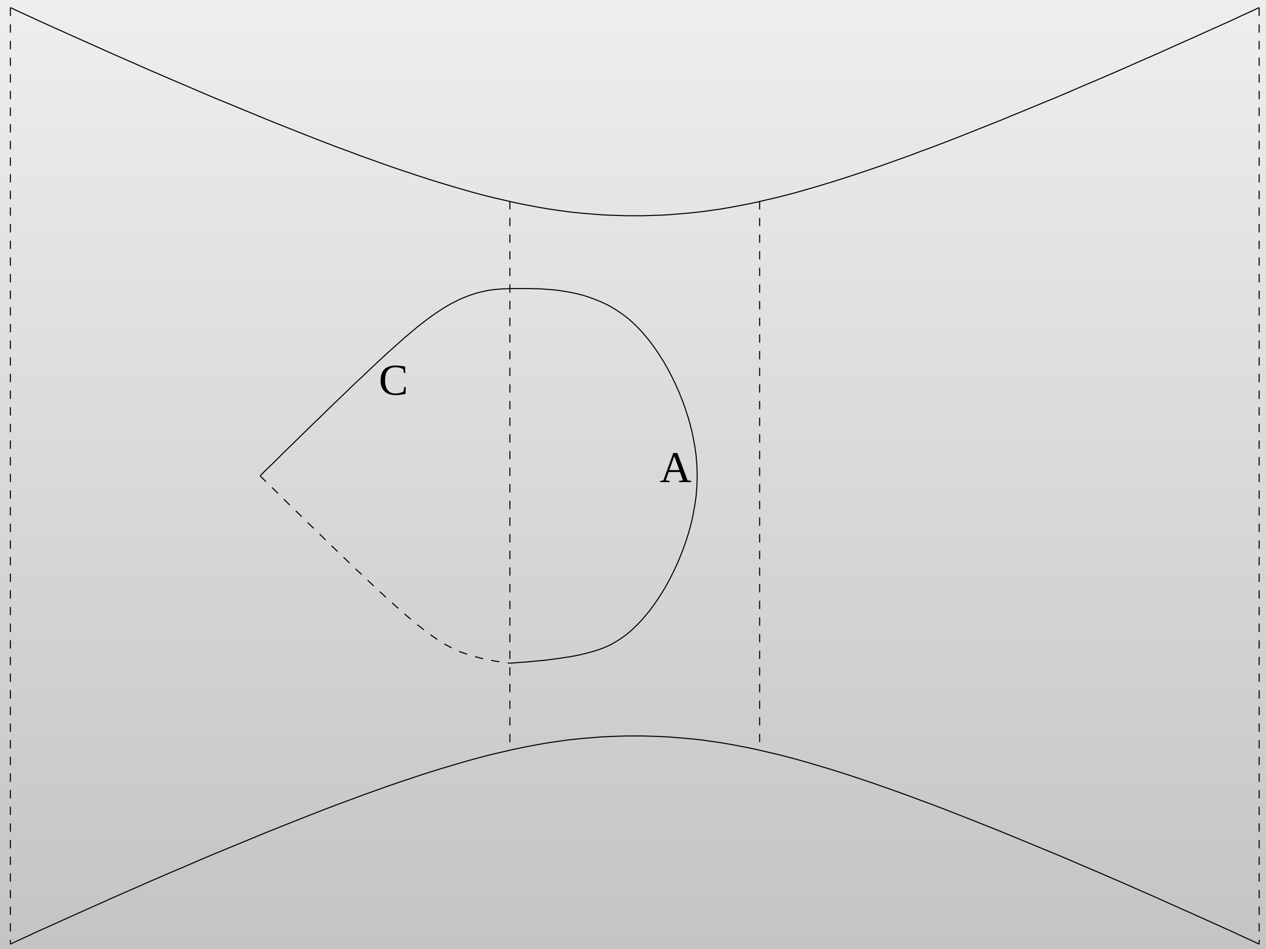
In some cases, canonical transformations can be used to absorb β in $\tilde{h}^{ab} := \beta h^{ab}$. [Tibrewala: arXiv:1311.1297]

Poisson brackets with structure functions can be interpreted as Lie algebroid. [Blohmann, Barbosa Fernandes, Weinstein: arXiv:1003.2857]
Brackets with $\beta \neq 1$ related to $\beta = 1$ by Lie algebroid morphism, as long as β is positive. [with F. D'Ambrosio]

Not isomorphic if β changes sign.
Signature change as new space(-time) structure.

Tricomi problem

$-\frac{\partial^2 u}{\partial t^2} + \beta(\mathcal{H})\Delta u = 0$: Characteristic C connected to arc A.



Need future data: no deterministic evolution.

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Conclusions I

Effective framework of loop quantum gravity being developed.

→ Minisuperspace-based loop quantum cosmology does not appear viable.

Motivation for minisuperspace models:
Hopefully “close” to full theory.

Possibility of signature change: Exact homogeneity does not reliably capture quantum space-time structure.

→ Useful for Wheeler–DeWitt quantum cosmology.

Or general scenarios of loop quantum cosmology at lower density, with inhomogeneity.

Significant difference between quantum-gravity models and quantized fields on a background.

Conclusions II

Recall that $\{\langle\widehat{\mathrm{pol}}_A\hat{C}_I\rangle, \langle\widehat{\mathrm{pol}}_B\hat{C}_J\rangle\}$ first class if $[\hat{C}_I, \hat{C}_J]$ first class.

Off-shell statement.
Independent of solutions to constraints or physical Hilbert space.

Reliable conclusions from effective constraint algebra possible, even if truncation to some $\hbar$-order provides poor approximation of dynamics.

Poor control on complete set of quantum corrections in loop quantum cosmology. (Higher-curvature terms?)

Nevertheless, effective constraints indicate the form of space(-time) structure realized.