A tale of cosmic doomsdays: from modified theories of gravity to quantum cosmology

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Outline

Outline

Introduction-1-: A brief sketch of the universe

- The universe is homogeneous and isotropic on large scales.
- The matter content of the universe:
  - Standard matter
  - Dark matter
  - Something that accelerates the universe (broadly speaking dark energy)
- What is dark energy (DE)? No idea. But we know it implies acceleration in a homogeneous and isotropic universe

Introduction-2-

- How do we know the universe is accelerating?
- Observational evidence
- How to describe this acceleration from and effective point of view?
  - Dark energy
  - Modified gravity
  - Other possibilities: Multiverse ...
- Our ignorance can be encoded on an effective equation of state

Introduction-3-

- Equation of state of dark energy is roughly -1
- Room for dark energy with $w_0 < -1 \Longrightarrow$ phantom energy
- In phantom energy models
  - Null energy condition is not satisfied
  - Energy density is a growing function of the scale factor (in an expanding Universe like ours)
  - May be a big rip singularity in the future

Starobinsky 00, Caldwell 02, Caldwell, Kamionkowski and Weinberg 03

Introduction 4: Phantom energy with $w = constant$

- Equation of state $p = w\rho$, $w = \mathrm{const.}$ and $w < -1$
- Energy density grows as a power of the scale factor
- Scale factor blows up in a finite future cosmic time
- The Hubble rate blows up in a finite cosmic time
- The cosmic time derivative of the Hubble rate does it too.
- Big rip in the future
- It was soon realised that there are more type of future singularities

Outline

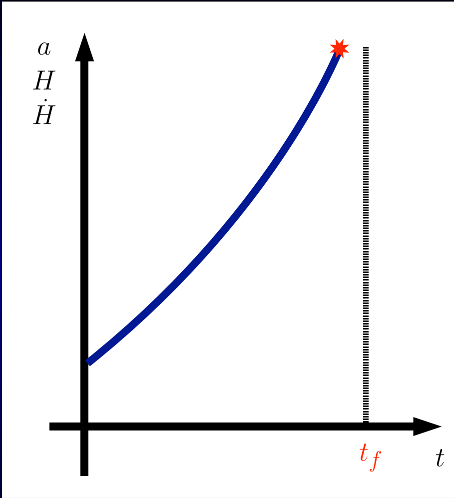
Cosmological singularities related to dark energy

- Classification of the cosmological singularities related to dark energy
  - Big rip singularity
  - Sudden singularity, big brake singularity, big démarrage singularity
  - Big freeze singularity
  - Type IV singularity
  - Little rip event
  - Little sibling of the big rip singularity

Kamenshchik 13 (review mainly on type II sing.)

Big rip singularity-1-

- For this singularity the null energy condition is violated. The scale factor diverges in a finite time. It is accompanied with a divergence of the Hubble rate and the cosmic derivative of the Hubble rate.



Starobinsky 00, Caldwell 02, Caldwell, Kamionkowski and Weinberg 03

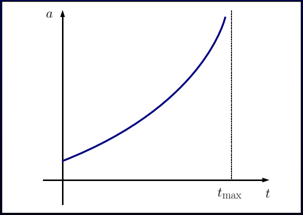
Big rip singularity-2-

- Equation of state $p = w\rho$, $w = \mathrm{const.}$ and $w < -1$
- Energy density $\rho = \tilde{A} a^{-3(w+1)}$
- Scale factor for a flat FLRW ($C = (\kappa_4^2/3)\tilde{A}$)

$$a(t) = \left[a_0^{3(w+1)/2} + \frac{3(w+1)}{2} C^{1/2}(t - t_0)\right]^{2/(3(w+1))}$$

- Big rip in the future

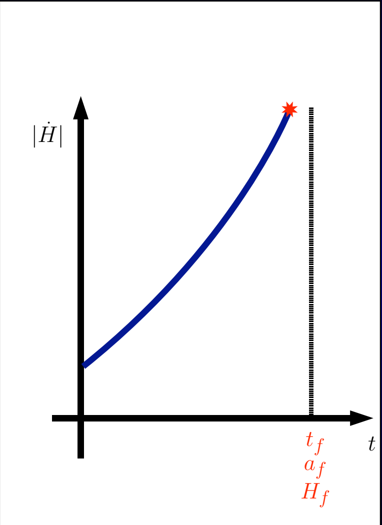
$$t_{\max} = t_0 - \frac{2}{3(w+1)C^{1/2}} a_0^{3(w+1)/2}$$



Sudden singularity

- This singularity occurs at a finite value of the scale factor and the Hubble rate. It is accompanied with a divergence of the cosmic derivative of the Hubble rate.

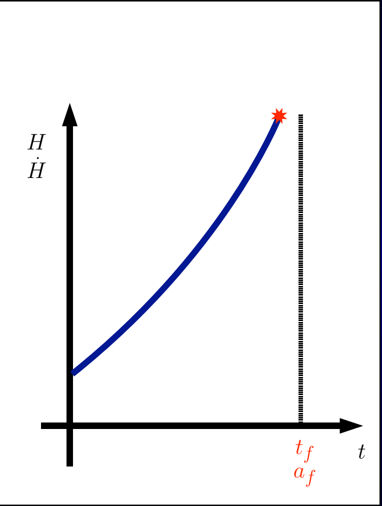
Barrow '04



Big freeze singularity

- This extremal events happens also at a finite scale factor. The Hubble rate and its cosmic derivative blow up at that scale factor.

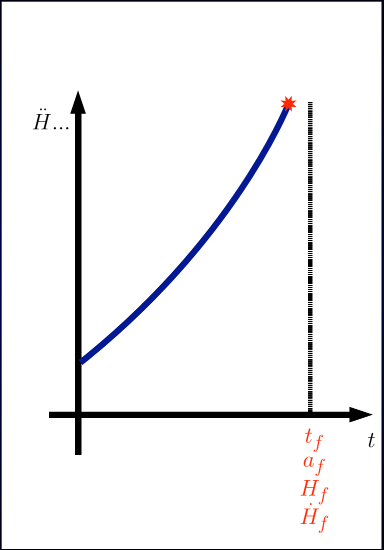
BL, González-Díaz and Martín-Moruno 06, 07



Type IV singularity

- None of the Hubble rate or $\dot{H}$ blow up in this case. However, second and higher derivatives blow up at a finite value of the scale factor.

Nojiri, Odintsov and Tsujikawa 05'



What time of matter can drive those singularities? Example: Generalised Chaplygin gas

- A sudden, big freeze and type IV singularity can emerges on the realm of a Chaplygin gas
- Chaplygin gas: $\alpha = 1$ and $A > 0$. Kamenshchik et al 01, Bilic et al 01
- Generalised Chaplygin gas: $P = -A/\rho^{\alpha}$, $0 < \alpha < 1$ and $A > 0$. Bento et al '02
- Morivated initially not only as a dark energy component but also a dark component playing the role of dark matter and dark energy

GCG and dark energy related singulaties-1-:

The asymptotic behaviour of a universe filled with each type of a "plain" GCG; i.e. it doesn't violate the null, strong and weak energy conditions

A,B	$1+\alpha$	a	ρ	Past	Future
$A<0$	positive	$0 \leq a < a_{\rm max}$	$0 \leq \rho < \infty$	dust-like	(1) no singularity/infinite future
					(2) type IV singularity
					(3) sudden singularity
$B>0$	negative	$a_{\rm min} \leq a < \infty$	$0 \leq \rho < \infty$	big freeze singularity	dust-like
$A>0$	$(2n)^{-1}>0$	$0 \leq a < a_{\rm max}$	$0 \leq \rho < \infty$	dust-like	no singularity/infinite future
$B<0$	$(2n)^{-1}<0$	$a_{\rm min} \leq a < \infty$	$0 \leq \rho < \infty$	big freeze singularity	dust-like

- (1) and (3) correspond to $-1<\alpha\leq -1/2$ and $0<\alpha$, respectively. (2) corresponds to $-1/2<\alpha<0$, where α cannot be expressed as $\alpha=1/(2p)-1/2$, with p a positive integer. If $-1/2<\alpha<0$ and α can be expressed as $\alpha=1/(2p)-1/2$, with p a positive integer, there is no past singularity and the universe is born at a finite past.

$$\rho = (A + B/a^{3(1+\alpha)})^{1/1+\alpha}$$

BL, González-Díaz, Martín-Moruno '06, '07

GCG and dark energy related singulaties-2-:

The asymptotic behaviour of a universe filled with each type of a phantom GCG

A,B	$1+\alpha$	a	ρ	Past	Future
$A>0$	positive	$a_{\min} \leq a < \infty$	$0 \leq \rho \leq A^{1/(1+\alpha)}$	(1) ∞ past	asymptotically dS
				(2) Type IV singularity	
				(3) Sudden singularity	
$B<0$	negative	$0 \leq a < a_{\max}$	$A^{1/(1+\alpha)} \leq \rho < \infty$	asymptotically dS/∞ past	big freeze singularity
$A<0$	$(2n)^{-1} > 0$	$a_{\min} \leq a < \infty$	$0 \leq \rho \leq \|A\|^{1/(1+\alpha)}$	∞ past	asymptotically dS
$B>0$	$(2n)^{-1} < 0$	$0 \leq a < a_{\max}$	$\|A\|^{1/(1+\alpha)} \leq \rho < \infty$	asymptotically dS/∞ past	big freeze singularity

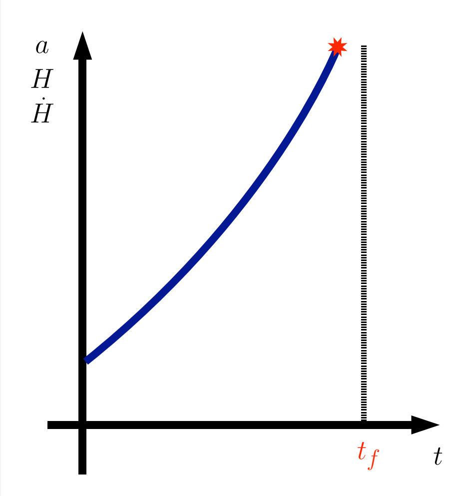
- (1) and (3) correspond to $-1 < \alpha \leq -1/2$ and $0 < \alpha$, respectively. (2) corresponds to $-1/2 < \alpha < 0$, where α cannot be expressed as $\alpha = 1/(2p) - 1/2$, with p a positive integer. If $-1/2 < \alpha < 0$ and α can be expressed as $\alpha = 1/(2p) - 1/2$, with p a positive integer, there is no past singularity and the universe is born at a finite past.

$$\rho = \left(A + B/a^{3(1+\alpha)}\right)^{1/1+\alpha}$$

BL, González-Díaz, Martín-Moruno '06, '07

Little rip singularity-1-

- For this singularity the null energy condition is violated. The scale factor diverges in an infinite time ($t_f \to \infty$). It is accompanied with a divergence of the Hubble rate and the cosmic derivative of the Hubble rate.



Little rip singularity-2-

- The name of little rip was introduced by Frampton, Ludwick and Scherrer '11
- This kind of singularity corresponds to a big rip sent towards an infinite cosmic time
- Examples:
  - This kind of singularity can happens in a FLRW universe filled with a perfect fluid $p = -\rho - A\rho^{1/2}$
    (Nojiri, Odintsov and Tsujikawa 05', Stefançic 05')
  - Also presents in some dilatonic brane-world models (BL 05').
  - First example was found by Ruzmaikina and Ruzmaiki back in 1970 corresponding to past little rip

Little sibling of the big rip singularity

- This event is much smoother than the big rip singularity. When the little sibling of the big rip is reached, the Hubble rate and the scale factor blow up but the cosmic derivative of the Hubble rate does not. This abrupt event takes place at an infinite cosmic time where the scalar curvature explodes.

BL, Errahmani, Martin-Moruno, Ouali, Tavakoli (2014)

Outline

Introduction-1-

- Einstein general relativity (GR) is an extremely successful theory for nearly a century
- However, it is expected to break down at some point at very high energies
- GR cannot explain the current acceleration of the universe unless a dark energy component is considered
- These are some motivations for looking for possible extension of GR
- There have been many proposals for alternative theories of GR as old as the theory itself
- One of the oldest proposal was due to Eddington

Introduction-2-

- In Eddington proposal, the connection rather than the metric plays the fundamental role of the theory
- It is equivalent to GR in vacuum
- BUT does not incorporate matter
- Recently an Eddington-inspired-Born-Infeld theory has been proposed by Bañados and Ferreira

Bañados and Ferreira (2010)

EiBI theory-1-

$$\mathcal{S}_{\rm EiBI}(g,\Gamma,\Psi)=\frac{2}{\kappa}\int d^4x\left[\sqrt{|g_{\mu\nu}+\kappa R_{\mu\nu}(\Gamma)|}-\lambda\sqrt{g}\right]+\mathcal{S}_{\rm m}(g,\Gamma,\Psi)$$

- We consider the action under the Palatini formalism, i.e., the connection $\Gamma^{\alpha}_{\mu\nu}$ is *not* the Levi-Civita connection of the metric $g_{\mu\nu}$
- This Lagrangian has two well defined limits: (i) when $|\kappa R|$ is very large, we recover Eddington's theory and (ii) when $|\kappa R|$ is small, we obtain the Hilbert-Einstein action with an effective cosmological constant $\Lambda=(\lambda-1)/\kappa$
- A solution of the above action can be characterized by two different Ricci tensors: $R_{\mu\nu}(\Gamma)$ as presented on the action and $R_{\mu\nu}(g)$ constructed from the metric g
- There are in addition three ways of defining the scalar curvature. These are: $g^{\mu\nu}R_{\mu\nu}(g)$, $g^{\mu\nu}R_{\mu\nu}(\Gamma)$ and $R(\Gamma)$. The third one is derived from the contraction between $R_{\mu\nu}(\Gamma)$ and the metric compatible with the connection Γ
- Therefore whenever one refers to singularity avoidance, one must specify the specific scalar curvature(s)

EiBI theory-2-

- Gravitational action:

$$\mathcal{S}_{\rm EiBI}(g,\Gamma,\Psi) = \frac{2}{\kappa}\int d^4x \left[\sqrt{|g_{\mu\nu}+\kappa R_{\mu\nu}(\Gamma)|}-\lambda\sqrt{g}\right]+\mathcal{S}_{\rm m}(g,\Gamma,\Psi)$$

- The parameter κ has been constrained using observationally for example from BBN (Casanellas et al 2012, Avelino 2012).
- The model can avoid the Big Bang singularity, for example, in a radiation dominated universe (Bañados and Ferreira 2012).
- Has been proposed as an alternative scenario to the inflationary paradigm (Avelino 2012)
- Not everything are nice news about EiBI theory (See for example Escamilla-Rivera et al 2012).
- Can this theory avoid the big rip singularity? This is the main question, we will address
- It was shown that if the null energy condition is fullfilled then the *apparent* null energy condition is also fullfilled (Deslate and Steinhoff 2012).

EiBI theory-3-

- The physical metric: flat FLRW metric with scale factor $a(t)$
- The auxiliary metric: $q_{\mu\nu} = -U(t)dt^2 + a^2(t)V(t)d\mathbf{X}^2$
- Friedmann eq: $H^2 = H^2(\kappa, \rho_t, p_t, \frac{dp_t}{d\rho_t})$
- The auxiliary metric: $U = U(\rho_t, p_t)$ and $V = V(\rho_t, p_t)$.
- The conservation of the energy momentum tensor holds.

EiBI theory-3-: radiation dominated universe

- A radiation dominated universe faces in the past a bounce ($\kappa < 0$) or a loitering effect ($\kappa > 0$); i.e. it avoid the big bang singularity. The reason behind this is that the energy density is bounded as a consequence of the modified Friedmann equation. Notice nevertheless the behaviour of the curvature as defined from the connection.
- The curvature behaviour (a_l, a_b minimum scale factors for κ positive and negative, respectively)

Curvature	Loitering ($\kappa > 0$)	Bounce ($\kappa < 0$)
$R_{00}(g)$	0	$4/\kappa$
$R_{ij}(g)$	0	$-4/(3\kappa)\, a_b^2$
$g^{\mu\nu}R_{\mu\nu}(g)$	0	$-8/\kappa$
$R_{00}(\Gamma)$	$1/\kappa$	∞
$R_{ij}(\Gamma)$	$-a_l^2\delta_{ij}/\kappa$	$-(1/\kappa)\, a_b^2\delta_{ij}$
$h^{\mu\nu}R_{\mu\nu}(\Gamma)$	$-\infty$	$+\infty$
$g^{\mu\nu}R_{\mu\nu}(\Gamma)$	$-4/\kappa$	$-\infty$

- Can the big rip be avoided as well?

The EiBI scenario filled with CDM and PE

- We consider the EiBI model filled with CDM and a dark energy component with a constant equation of state $w \sim -1$.
- In GR a matter component such that $w \preceq -1$ (and constant) implies a big rip singularity. Can the EiBI scenario avoid this singularity as happens with the big bang (with respect to the metric $g_{\mu\nu}$) singularity?

Curvature	Big Freeze ($\kappa < 0$)	Big Rip $\kappa > 0$
$R_{00}(g)$	$-\infty$	$-\infty$
$R_{ij}(g)$	$+\infty$	$+\infty$
$g^{\mu\nu}R_{\mu\nu}(g)$	$+\infty$	$+\infty$
$R_{00}(\Gamma)$	finite	$-\infty$
$R_{ij}(\Gamma)$	finite	$+\infty$
$h^{\mu\nu}R_{\mu\nu}(\Gamma)$	finite	$4/\kappa$
$g^{\mu\nu}R_{\mu\nu}(\Gamma)$	finite	$+\infty$

- Notice that EiBI reduce to GR at late-time for a dust filled universe this is no longer the case for a universe filled with phantom matter. Nevertheless the big rip singularity is not cured BL, Chen and Chen 13

The EiBI scenario and the big rip-1-

- The cosmic time elapsed from the present time to the Big Rip singularity time, normalized to the current Hubble parameter; i.e., $H_0(t_{\rm sing} - t_0)$, for different values of ϵ ($w = -(1+\epsilon)$) in GR and in the EiBI theory. We see that such cosmic time remains finite in the EiBI theory, meaning that the Big Rip singularity is inevitable. Here we assume $\Omega_m = 0.277$ and $\Omega_w = 0.728$ (WMAP9)

ϵ	$H_0(t_{\rm sing} - t_0)$(GR)	$H_0(t_{\rm sing} - t_0)$(EiBI)
0.02	37.14	37.19
0.04	18.99	19.04
0.06	12.74	12.80
0.08	9.58	9.63
0.10	7.67	7.73
0.12	6.40	6.45
0.14	5.26	5.31

The EiBI scenario and the big rip-2-

- The cosmic time elapsed from the present time to the Big Rip singularity time, normalized to the current Hubble parameter; i.e., $H_0(t_{\rm sing} - t_0)$, for different values of ϵ ($w = -(1+\epsilon)$) in GR and in the EiBI theory. We see that such cosmic time remains finite in the EiBI theory, meaning that the Big Rip singularity is inevitable. Here we assume $\Omega_m = 0.315$ and $\Omega_w = 0.690$ (Planck)

ϵ	$H_0(t_{\rm sing} - t_0)$(GR)	$H_0(t_{\rm sing} - t_0)$(EiBI)
0.02	38.14	37.19
0.04	19.50	19.55
0.06	13.08	13.13
0.08	9.83	9.89
0.10	7.87	7.93
0.12	6.56	6.61
0.14	5.39	5.44

What about the other singularities?

Singularity in GR	EiBI physical metric	EiBI auxiliary metric
Big Rip	Big Rip	expanding de-Sitter
past Sudden ($\alpha > 0$)	past Type IV ($0 < \alpha \leq 2$)	contracting de-Sitter
	past Sudden ($\alpha > 2$)	
future Big Freeze ($\alpha < -1$)	future Big Freeze ($-3 < \alpha < -1$)	expanding de-Sitter
	future Type IV ($\alpha = -3$)	
	future Sudden ($\alpha < -3$)	
past Type IV ($-1 < \alpha < 0$) ($\alpha \neq -n/(n+1)$)	past Sudden ($-2/3 < \alpha < -1/3$)	past Type IV
	(1)past Type IV	
	(2)finite past without singularity	finite past without singularity
	past loitering effect ($a_b > a_{\min}$)	Big Bang
finite past without singularity ($\alpha = -n/(n+1)$) ($-1 < \alpha < 0$)	finite past without singularity	finite past without singularity
	past loitering effect ($a_b > a_{\min}$)	Big Bang
Little Rip	Little Rip	expanding de-Sitter

BL, Chen and Chen 14

The geodesic analyses of a Newtonian object in the EiBI setup-1-

- A spherical Newtonian object with mass M and a test particle rotating around the object with a physical radius r
- Both of them are embedded in a spherically symmetric FLRW background
- We will analyse the fate of the bound structure near the singularities corresponding to the physical metric and the auxiliary metric
- The evolution equation of the physical radius: $\ddot{r} = \frac{\ddot{a}}{a} r - \frac{GM}{r^2} + \frac{L^3}{r_3}$ conservation of angular momentum: $r^2 \dot{\phi} = L$
- Near the Big Rip, Little Rip, Big Freeze and the Sudden singularities: $\ddot{r} \approx \frac{\ddot{a}}{a} r$
- $r_1 = a(t)$, and $r_2 = r_1 \int \frac{dt}{{r_1}^2}$
- $r(t) = A_1 r_1(t) + A_2 r_2(t)$

Faraoni, Jacques 2007

The geodesic analyses of a Newtonian object in the EiBI setup-2-

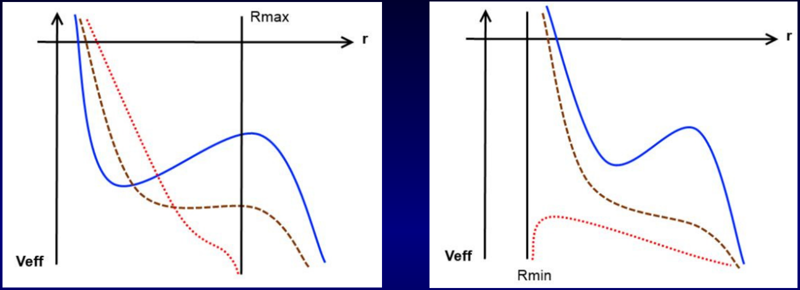


Figure: We show the behaviour of the effective potential *Veff* ($\dot{r}^2 = -2V_{\rm eff}$) for future singularities (left figure) and past singularities (right figure). $R_{\rm max}$ is finite for a sudden and big freeze singularities while infinite for a big rip and little rip singularity. Likewise $R_{\rm min}$ is finite for a past sudden singularity. On the left figure: the blue solid curve shows the current bound structure, the brown dashed one the intermediate future behaviour and the red dotted one the final state. On the right figure the colors appear in an inverted chronological order, first red dotted, then brown dashed and finally blue solid, as the singularity takes place in the past.

The geodesic analyses of a Newtonian object in the EiBI setup-3-: Type IV singularity and the geodesic defined by the auxiliary metric

- Near a type IV singularity, all the terms in the evolution equation are finite
- A bound system remains bounded
- As for the geodesic equations defined by the auxiliary metric, the singularities are substituted by a de-Sitter or a type IV
- the auxiliary metric and the physical connection have a much smoother behaviour close to the singularities

Outline

On the quantum fate of singularities in a dark-energy dominated universe

Within the framework of quantum geometrodynamics

- It was shown by Dabrowski, Kiefer and Sandhöfer 06' that this kind of singularity can be removed in the context of the Wheeler de Witt Eq./formalism; i.e. in the framework of quantum geometrodynamics.
- It was shown by Kamenshchik, Kiefer and Sandhöfer 07' the avoidance of a big brake singularity.
- It was shown by BL, Kiefer, Sandhöfer and Vargas Moniz 09' the avoidance of a big démarrage singularity and a big freeze.
- Type IV singularity are partially removed (BL, Krämer and Kiefer 2014).

Outline

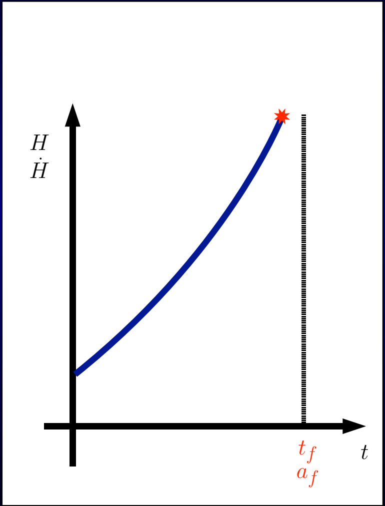
The big freeze singularity with phantom matter

- The generalized Chaplygin gas (Kamenshchik et al '01, Bento et al '02) ($0 < A$, $B < 0$, $\beta < -1$)

$$P = -A/\rho^{\beta} \Longrightarrow$$

$$\rho = \left(A + \frac{B}{a^{3(1+\beta)}}\right)^{\frac{1}{1+\beta}}$$

- Phantom GCG $\Longrightarrow P + \rho < 0$
- At $a_{\max} = a_f$: The energy density ρ, the Hubble rate H, $\dot{H}$ $\to \infty$. The pressure $P \to -\infty$
- BF singularity in the future



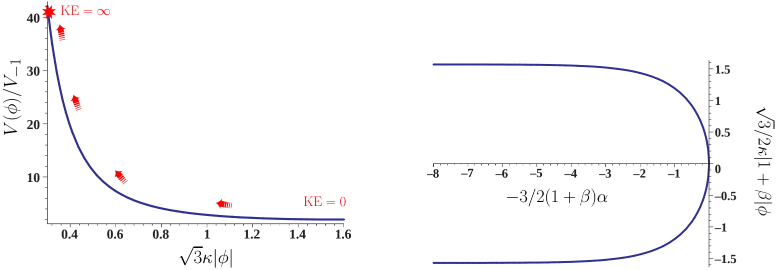
The BF singularity driven by a phantom scalar field

- Phantom scalar field ϕ: $\rho_\phi = -\frac{1}{2}\dot{\phi}^2 + V(\phi)$, $p_\phi = -\frac{1}{2}\dot{\phi}^2 - V(\phi)$

$$\ddot{\phi} + 3H\dot{\phi} - V'(\phi) = 0, \quad V(\phi) \simeq V_{-1}\left(\frac{\sqrt{3}}{2}\kappa|1+\beta||\phi|\right)^{-\frac{2\beta}{1+\beta}}$$

- Identify ϕ and GCG; i.e. $\rho_\phi = \rho$, $p_\phi = P$

$$V_{-1} = A^{1/(1+\beta)}/2\,, \quad \alpha = \ln(a/a_{\rm max})$$



BL, Kiefer, Sandhöfer, Vargas Moniz, 09

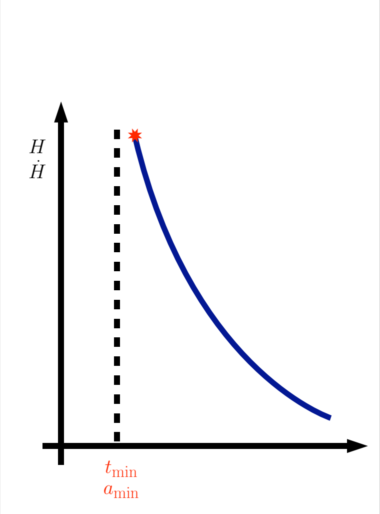
The big freeze singularity with standard matter

- The generalized Chaplygin gas ($A < 0$, $0 < B$, $\beta < -1$)

$$P = -A/\rho^{\beta} \Longrightarrow$$

$$\rho = \left(A + \frac{B}{a^{3(1+\beta)}}\right)^{\frac{1}{1+\beta}}$$

- At $a_{\rm min} = a_f$: The energy density ρ, the Hubble rate H, the pressure $P \to \infty$ and $\dot{H} \to -\infty$
- BF singularity in the past



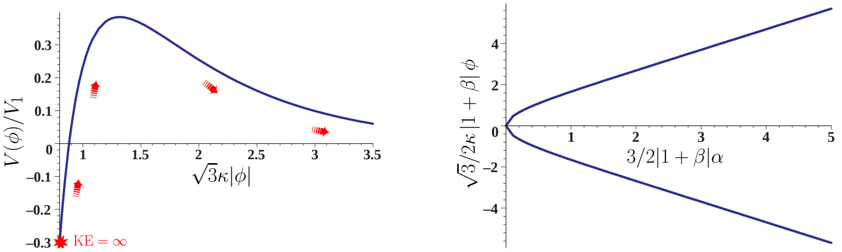
The BF singularity driven by a standard scalar field

- Standard scalar field ϕ: $\rho_\phi = +\frac{1}{2}\dot{\phi}^2 + V(\phi)\ , \quad p_\phi = +\frac{1}{2}\dot{\phi}^2 - V(\phi)$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad V(\phi) \simeq -V_1 \left(\frac{\sqrt{3}}{2}\kappa|1+\beta||\phi|\right)^{-\frac{2\beta}{1+\beta}}$$

- Identify ϕ and GCG; i.e. $\rho_\phi = \rho$, $p_\phi = P$

$$V_1 = |A|^{1/(1+\beta)}/2\,, \quad \alpha = \ln(a/a_{\min})$$



The Wheeler-DeWitt equation

- Quantisation of the classical scenario in the quantum geometrodynamical framework
- The Wheeler-DeWitt equation in quantum cosmology is the analogous to Schrödinger equation in quantum mechanics.
- The Wheeler-DeWitt equation for the space variables (a,ϕ)

$$\frac{\hbar^2}{2}\left(\frac{\kappa^2}{6}\frac{\partial^2}{\partial\alpha^2}-\ell\frac{\partial^2}{\partial\phi^2}\right)\Psi\left(\alpha,\phi\right)+a_0^6 e^{6\alpha}V(\phi)\Psi\left(\alpha,\phi\right)=0,\ \ \alpha:=\ln\left(\frac{a}{a_0}\right)$$

  - Standard scalar field $\ell=1$, $a_0=a_{\rm min}$
  - Phantom scalar field $\ell=-1$, $a_0=a_{\rm max}$
- Notice that in the quantum case ϕ is no longer a function of a
- General remark: the Wheeler-DeWitt equation does not depend on time (!)

Decomposing the Wheeler-DeWitt equation

- We use the ansatz

$$\Psi(\alpha,\phi)=\varphi_k(\alpha,\phi)C_k(\alpha)$$

- We require the matter part of the Wheeler-DeWitt equation to satisfy

$$-\ell\frac{\hbar^2}{2}\frac{\partial^2\varphi_k}{\partial\phi^2}+a_0^6e^{6\alpha}V(\phi)\varphi_k=E_k(\alpha)\varphi_k$$

- Such a Born–Oppenheimer-type of ansatz was first used in quantum cosmology in Kiefer 88
- Schrödinger type of equation and in the vicinity the singularity reads

$$\varphi_k''+\left[\ell k^2+\tilde{V}_\alpha|\phi|^{-\frac{2\beta}{1+\beta}}\right]\varphi_k=0$$

where $k^2:=\frac{2E_k}{\hbar^2}$, $\tilde{V}_\alpha:=\frac{2V_\alpha}{\hbar^2}$

$$V_\alpha:=a_0^6e^{6\alpha}V_\ell\left[\frac{\sqrt{3}\kappa}{2}|1+\beta|\right]^{-\frac{2\beta}{1+\beta}}$$

Singular potentials-1-

- The matter part of the wave function satisfies

$$\varphi_k'' + \left[\ell k^2 + \tilde{V}_\alpha |\phi|^{-\frac{2\beta}{1+\beta}}\right] \varphi_k = 0$$

- This equation is formally the same as the radial part of the stationary Schrödinger equation for an *attractive* potential of inverse power $V \sim r^{-\frac{2\beta}{1+\beta}}$, where $|\phi|$ plays the role of the radial coordinate r, and the angular momentum vanishes.
- The potential corresponds to a singular potential; i.e. a potential that approaches (plus or minus) infinity faster than r^{-2} for $r \to 0$. For an attractive r^{-2}-potential there exists a transitional case: if the coupling is more negative than a critical value, the potential is singular, otherwise regular.

Singular potentials-2-

- Analytical solutions for polynomial singular potentials are known for the inverse square, inverse fourth-power, and inverse sixth-power potentials.
  - The inverse square potential is realized for $\beta \ll -1$, where β is chosen such that $|1+\beta||\phi|$ is still small
  - The inverse fourth-power potential corresponds to $\beta = -2$
  - The inverse sixth-power potential corresponds to $\beta = -\frac{3}{2}$
- We focus on the case $\beta \ll -1$. We thus deal with the case of the inverse-square potential $\frac{\tilde{V}_\alpha}{|\phi|^2}$ with

$$\tilde{V}_\alpha = \frac{2a_0^6 e^{6\alpha} V_\ell}{\hbar^2} \left[\frac{\sqrt{3}\kappa|\beta|}{2}\right]^{-2} > 0$$

- This case is sufficiently generic to accommodate also the features of other singular potentials.

More on the matter part of the wave function

- For the least singular potential, which is realized for $\beta \ll -1$, we have to solve the equation

$$\varphi_k'' + \left[\ell k^2 + \frac{\tilde{V}_\alpha}{|\phi|^2}\right]\varphi_k = 0, \quad \tilde{V}_\alpha = \frac{2a_0^6 e^{6\alpha} V_\ell}{\hbar^2}\left[\frac{\sqrt{3}\kappa|\beta|}{2}\right]^{-2} > 0$$

- The phantom and scalar matter have to obey the same quantum equation, where the realm of positive energy for the ordinary scalar field $k^2 > 0$ corresponds to the realm of negative energy for the case of the phantom field, $k^2 < 0$
- The general solution is

$$\varphi_k(\alpha, |\phi|) = \sqrt{|\phi|}\left[c_1 \mathrm{J}_\nu(\sqrt{\ell}k|\phi|) + c_2 \mathrm{Y}_\nu(\sqrt{\ell}k|\phi|)\right], \ \nu := \sqrt{\frac{1}{4} - \tilde{V}_\alpha}$$

- There are four cases to distinguish: k can be real or imaginary, depending on whether the energy entering k^2 is positive or negative. Furthermore, ν can be real or imaginary, depending on the parameters β, A, and the value of α

The gravitational part of the wave function

- The gravitational part of the wave function fulfils

$$\frac{\kappa^2}{6}\left(2\dot{C}_k\dot{\varphi}_k + C_k\ddot{\varphi}_k\right) + \left(\frac{\kappa^2}{6}\ddot{C}_k + k^2 C_k\right)\varphi_k = 0$$

- The Born–Oppenheimer approximation: $\dot{C}_k\dot{\varphi}_k$ and $C_k\ddot{\varphi}_k$ can be neglected.
  - C_k varies much more rapidly with α than φ_k
  - Neglect the back reaction of the matter part on the gravitational part
  - The change in the matter part does not influence the gravitational part
- The matter part simply contributes its energy through k^2

$$\left(\frac{\kappa^2}{6}\ddot{C}_k + k^2 C_k\right)\varphi_k = 0 \Longrightarrow C_k(\alpha) = b_1 e^{\mathrm{i}\frac{\sqrt{6}k}{\kappa}\alpha} + b_2 e^{-\mathrm{i}\frac{\sqrt{6}k}{\kappa}\alpha}$$

- Same solution for a phantom or a scalar field.

The wave function at the singularity

- The singularity occurs in the two models at $\phi = 0$ and $\alpha = 0$
- For $\alpha = 0$, $\nu := \nu_0 = \sqrt{\frac{1}{4} - \tilde{V}_{\alpha=0}}$. There are 3 cases:
  - $\frac{1}{4} - \tilde{V}_{\alpha=0} > 0 \Longrightarrow \nu_0$ is real and $0 < \nu_0 < \frac{1}{2}$
  - $\frac{1}{4} - \tilde{V}_{\alpha=0} < 0 \Longrightarrow \nu_0$ is imaginary
  - $\frac{1}{4} - \tilde{V}_{\alpha=0} = 0 \Longrightarrow \nu_0 = 0$
- The behaviour of the matter part of the wave function will depends on the value acquired by ν
- It can be shown that the matter part of the wave function always vanishes at $\phi = 0$. Notice that we have not used any boundary condition

Ex. of the matter part of the wave function at $\phi = 0$

- The matter part of the wave function

$$\varphi_k(\alpha, |\phi|) = \sqrt{|\phi|}\left[c_1 \mathrm{J}_\nu(\sqrt{\ell}k|\phi|) + c_2 \mathrm{Y}_\nu(\sqrt{\ell}k|\phi|)\right]$$

- For this example we restrict to $\frac{1}{4} - \tilde{V}_{\alpha=0} > 0$; i.e. $0 < \nu < \frac{1}{2}$
- Near the singularity, we can use

$$\mathrm{J}_\nu(z) \approx \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}, \qquad \nu \neq -1, -2, -3\ldots,$$

$$\mathrm{Y}_\nu(z) \approx -\frac{1}{\pi}\left(\frac{z}{2}\right)^{-\nu}\Gamma(\nu), \qquad \mathrm{Re}(\nu) > 0$$

- Then, the matter part of the wave function vanishes at $\phi = 0$
- It can be shown that this is true for the other two cases

Gravitational part of the wave function at $\alpha = 0$

- What does it mean that the wave function vanishes at the singularity? Singularity avoidance but not yet we have to make sure that the gravitational part of the wave function is bounded at the singularity
- The gravitational part of the wave function

$$C_k(\alpha) = b_1 e^{\mathrm{i}\frac{\sqrt{6}k}{\kappa}\alpha} + b_2 e^{-\mathrm{i}\frac{\sqrt{6}k}{\kappa}\alpha}$$

- Note that for $k^2 < 0$, k becomes imaginary and the dependence on α becomes exponential. In any case, $C_k(\alpha = 0) < \infty$, so the wave function remains finite at the respective singularities and we can safely speak of singularity avoidance.
- Finally, as the wave function vanishes, we can interpret this as a singularity avoidance.
- What about if we impose some boundary condition?

A Wise Boundary condition for the wave function-1

- Nobody knows what the correct boundary condition for the quantum universe are.
- There have been several proposals, most of them using the boundary condition with the ambition to lead to singularity avoidance
- We impose the BC: The wave function decreases in the classically forbidden region.
- Why? Because then it is possible to construct wave packets that follow classical trajectories with turning point in configuration space.
- Namely, one has to require that the wave packet decays in the classically forbidden region. This allows the interference of wave packets following the two branches of the classical solution behind the classical turning point.
- In general, out of solutions to the Wheeler–DeWitt equation which grow in the classically forbidden region, no wave packet can be constructed that follows the classical path

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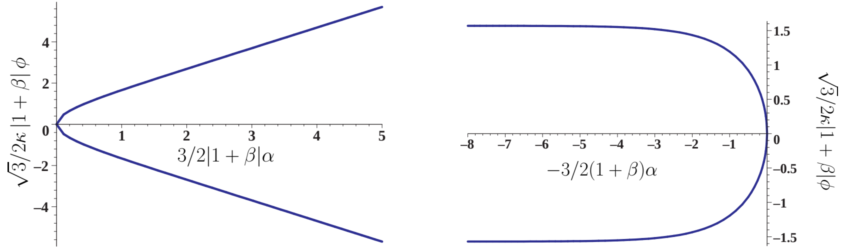
A Wise Boundary condition for the wave function-2

- How do we impose this boundary conditions?



Standard scalar field(p) Phantom scalar field(f)

Imposing the BC: standard scalar field-1-

- The region $a < a_{\rm min}$ is a forbidden region $\Longrightarrow$ we impose the boundary condition that the wave function decay there.
- Then, $\Psi \to 0$ as $\alpha \to -\infty$. The total wave function has to vanish well inside the forbidden region. This happens whenever the matter (or gravitational part) vanishes while the other part is bounded.
- When $\alpha \to -\infty$, $\nu \to \frac{1}{2}$, the matter-dependent part reads

$$\lim_{\alpha \to -\infty} \varphi_k(\alpha, |\phi|) = \sqrt{\frac{2}{\pi}} \left[c_1 \sin(k|\phi|) - c_2 \cos(k|\phi|)\right]$$

- This vanishes for small $|\phi|$ if $c_2 = 0$. Finally, the matter part of the wave function vanishes if

$$\varphi_k(\alpha, |\phi|) = c_1 \sqrt{|\phi|} \mathrm{J}_\nu(k|\phi|)$$

- What about the gravitational part? Is it bounded well inside the forbidden region?

Imposing the BC: standard scalar field-2-

- For positive energy, $k^2 > 0$, the gravitational part of the wave function is oscillating and the full solution is given by

$$\Psi_k(\alpha,\phi) = c_1\sqrt{|\phi|}\mathrm{J}_\nu(k|\phi|)\left[b_1 e^{\mathrm{i}\frac{\sqrt{6}k}{\kappa}\alpha} + b_2 e^{-\mathrm{i}\frac{\sqrt{6}k}{\kappa}\alpha}\right]$$

- For $k^2 < 0$, the gravitational part becomes exponential. To ensure that the boundary condition $\Psi \to 0$ as $\alpha \to -\infty$ is satisfied for the entire wave function, we have to set $b_1 = 0$. Thus, for imaginary k the gravitational part of the wave function decays exponentially for $\alpha \to -\infty$ whereas the matter part remains finite $\Longrightarrow$ The gravitational part alone ensures in this way that the wave function vanishes as $\alpha \to -\infty$. No additional condition arises for φ_k. The full solution for imaginary ($k \to \mathrm{i}k$) is thus

$$\Psi_k(\alpha,\phi) = b_2 e^{\frac{\sqrt{6}k}{\kappa}\alpha}\left[c_1\mathrm{J}_\nu(\mathrm{i}k|\phi|) + c_2\mathrm{Y}_\nu(\mathrm{i}k|\phi|)\right]$$

Imposing the BC: phantom scalar field

- The region $a > a_{\max}$ is a forbidden region $\Longrightarrow$ we impose the boundary condition that the wave function decay there.
- Then, $\Psi \to 0$ as $\alpha \to \infty$. The total wave function has to vanish well inside the forbidden region. This happens whenever the matter (or gravitational part) vanishes while the other part is bounded.
- The physical solutions are

$$\begin{aligned}
\Psi_k(\alpha,\phi) &= \left(b_1 e^{\mathrm{i}\frac{\sqrt{6}k}{\kappa}\alpha} + b_2 e^{-\mathrm{i}\frac{\sqrt{6}k}{\kappa}\alpha}\right)\sqrt{|\phi|}\mathrm{K}_{i\nu}(k|\phi|), \quad k^2 > 0 \\
\Psi_k(\alpha,\phi) &= d_2 \exp\left(-\frac{\sqrt{6}}{\kappa}\tilde{k}\alpha\right)\sqrt{|\phi|}\mathrm{H}^{(2)}_{\mathrm{i}\nu}(\tilde{k}|\phi|), \qquad k^2 < 0.
\end{aligned}$$

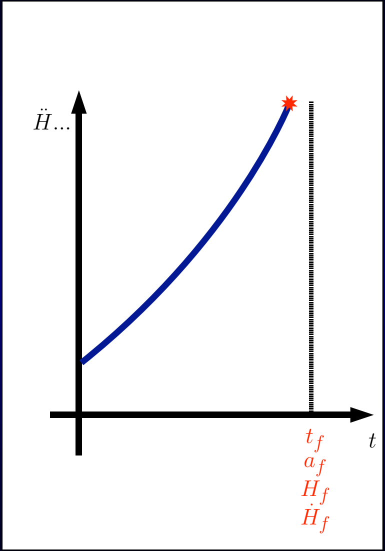
Outline

The type IV singularity with phantom matter

- The generalized Chaplygin gas ($0 < A$, $B < 0$, $-1/2 < \beta < 0$, $\beta \neq 1/(2p) - 1/2$)

$$P = -A/\rho^{\beta} \Longrightarrow$$
$$\rho = \left(A + \frac{B}{a^{3(1+\beta)}}\right)^{\frac{1}{1+\beta}}$$

- Phantom GCG $\Longrightarrow$ $P + \rho < 0$
- At $a_{\min} = a_f$: higher derivative of H blows up
- Type IV singularity in the past



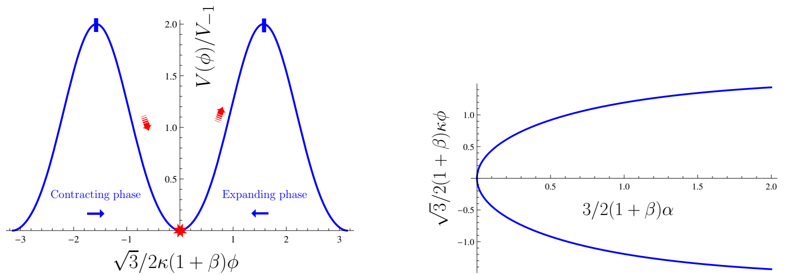
The type IV singularity driven by a phantom scalar field

- Phantom scalar field ϕ: $\rho_\phi = -\frac{1}{2}\dot{\phi}^2 + V(\phi)$, $p_\phi = -\frac{1}{2}\dot{\phi}^2 - V(\phi)$

$$\ddot{\phi} + 3H\dot{\phi} - V'(\phi) = 0, \quad V(\phi) \simeq V_{-1}\left(\frac{\sqrt{3}}{2}\kappa|1+\beta||\phi|\right)^{-\frac{2\beta}{1+\beta}}$$

- Identify ϕ and GCG; i.e. $\rho_\phi = \rho$, $p_\phi = P$

$$V_{-1} = A^{1/(1+\beta)}/2\,, \quad \alpha = \ln(a/a_{\rm min})$$



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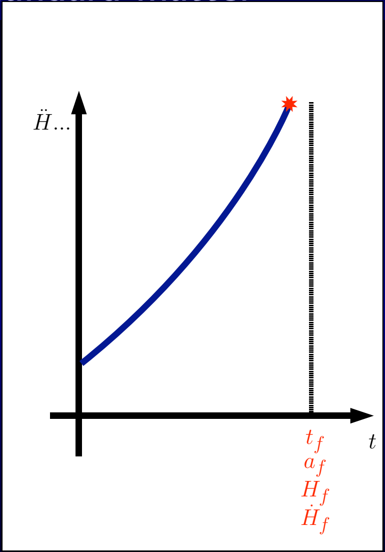
The type IV singularity with standard matter

- The generalized Chaplygin gas $(A < 0,\ 0 < B,\ -1/2\beta < 0)$

$$P = -A/\rho^{\beta} \Longrightarrow$$
$$\rho = \left(A + \frac{B}{a^{3(1+\beta)}}\right)^{\frac{1}{1+\beta}}$$

- At $a_{\max} = a_f$: Higher derivative of the Hubble rate blows up
- Type IV singularity in the future



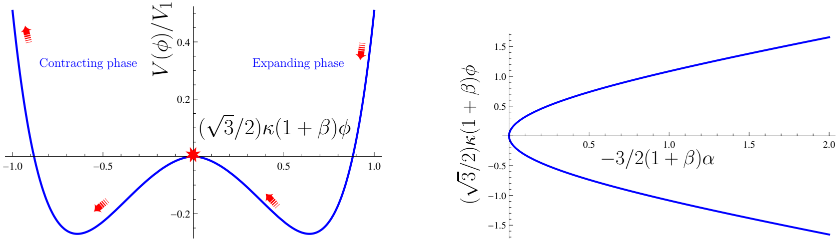
The BF singularity driven by a standard scalar field

- Standard scalar field ϕ: $\rho_\phi = +\frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \quad p_\phi = +\frac{1}{2}\dot{\phi}^2 - V(\phi)$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad V(\phi) \simeq -V_1\left(\frac{\sqrt{3}}{2}\kappa|1+\beta||\phi|\right)^{\frac{2\beta}{1+\beta}}$$

- Identify ϕ and GCG; i.e. $\rho_\phi = \rho$, $p_\phi = P$

$$V_1 = |A|^{1/(1+\beta)}/2\,, \qquad \alpha = \ln(a/a_{\rm max})$$



The quantum analysis of type IV singularity

- We follow a Born-Oppenheimer (BO) approximation
- We can solve exactly the matter part in same cases $\beta \rightarrow -1/2$ and $l = \pm 1$: it involves Heun functions.
- The gravitational part can be as well be solved within a WKB approximation
- Singularity avoidance for type IV singularities occurs only in special cases. In general, the singularity is not avoided; i.e. only a subset of the solutions of the Wheeler DeWitt equation vanishes at the singularity.

Outline

1. Introduction
2. Cosmological singularities related to dark energy
3. Smoothing DE singularities through a modification of gravity?
4. The quantum fate of singularities in a dark-energy dominated universe
   - Example 1: The quantum fate of the big freeze
   - Example 2: The quantum fate of type IV singularity
5. Conclusions

Conclusions

- In this talk, we have reviewed some of the cosmological singularities that have appeared on the literature over the last few years, motivated (initially) from the possible presence of an exotic dark energy component
- Then we have shown how these singularities could be removed or appeased either through some modified theories of gravity or within a quantum approach
- We have chosen the EiBI theories as an example of a modified theory of gravity
- The Quantum approach has been carried out in the quantum geometrodynamics setup and within the BO approaches