Cosmology of multiscale spacetimes

based on

JCAP 12 (2013) 041 [arXiv:1307.6382]

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July 29th, 2014

Outline

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01/33– Dimensional flow in quantum gravity

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• Noncommutative geometry [Connes 2006; Benedetti 2008; Alesci & Arzano 2012]; • CDT [Ambjørn, Jurkiewicz & Loll 2005; Benedetti & Henson 2009]; • Spin foams [Modesto (et al.) 2008–10]; • AS [Lauscher & Reuter 2005; Reuter & Saueressig 2011; G.C. et al. 2013]; • HL gravity [Hořava 2008,2009].

02/33– Field theory on multiscale spacetimes

- Formalism describing this and other features of QG theories with an alternative toolbox from multifractal geometry, transport and probability theory, complex systems. (Exceptions in ordinary HEP are the rule.)

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- Aims: (i) to see how anomalous geometries affect the physics; (ii) to capture regimes in QG; (iii) to apply the same tools in QG.
- Various works in 2009-2013 (also with M. Arzano, A. Eichhorn, J. Magueijo, G. Nardelli, D. Oriti, D. Rogríguez, F. Saueressig, M. Scalisi).

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3. Work out dynamics with usual variational principle and techniques.
4. Differs from ST theories (v is not a Lorentzian scalar field) and unimodular gravity (metric structure is fully dynamical).

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4. Cosmology: bounce and alternatives to inflation, big-bang and Λ problems reinterpreted.

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- Action measure $\mathrm{d}\varrho(x) = \mathrm{d}^D x\, v(x) = \mathrm{d}^D q(x)$. General factorizable measures:

$$v(\ell_n, x) = \prod_{\mu=0}^{D-1} v_\mu(\ell_n, x^\mu)\,, \qquad v_\mu(\ell_n, x^\mu) \geq 0\,.$$

06/33– Example 1: Fractional measure

Represents random fractals.

$$d\varrho_\alpha(x) = d^D x\, v_\alpha(x) = d^D x \prod_\mu \frac{|x^\mu|^{\alpha-1}}{\Gamma(\alpha)}$$

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'Geometric' coordinates:

$$q^\mu := \varrho_\alpha(x^\mu) = \frac{\mathrm{sgn}(x^\mu)|x^\mu|^\alpha}{\Gamma(\alpha+1)} \qquad \Rightarrow \qquad \mathrm{d}\varrho_\alpha = \mathrm{d}^D q$$

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Scaling property:

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Same result obtained via self-similarity theorem or via operational definition as the scaling of the volume of a D-ball of radius R: $\mathcal{V}^{(D)}(R) = \int_{D\text{-ball}} \mathrm{d}\varrho_\alpha(x) \propto R^{D\alpha}$.

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Scale-dependent Hausdorff dimension. Simplest (but not toy) model, two terms (binomial measure):

$$I_D = I_D^{\alpha_1} + \ell_*^{D(\alpha_1-\alpha_2)} I_D^{\alpha_2} , \qquad [I_D] = -D\alpha_1 , \qquad \tfrac{1}{2} \leq \alpha_1 < \alpha_2 \leq 1 .$$

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$$\mathcal{V}^{(D)}(R) = \ell_*^{D\alpha_1}\left[\Omega_{D,\alpha_1}\left(\frac{R}{\ell_*}\right)^{D\alpha_1} + \Omega_{D,\alpha_2}\left(\frac{R}{\ell_*}\right)^{D\alpha_2}\right].$$

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$R \ll \ell_*$: $\mathcal{V}^{(D)} \sim R^{D\alpha_1}$

$R \gg \ell_*$: $\mathcal{V}^{(D)} \sim \tilde{R}^{D\alpha_2}$, $\tilde{R} = R\ell_*^{-1+\alpha_1/\alpha_2}$

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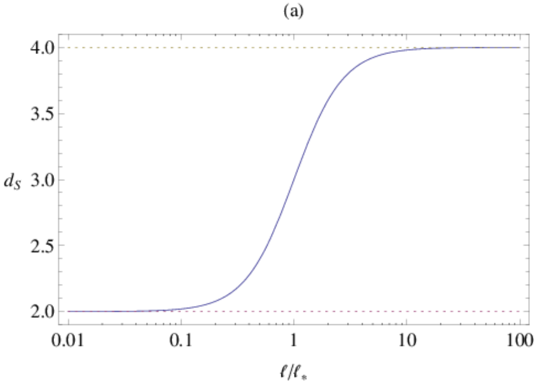

09/33– Example 3: Log-oscillating measure

$$\varrho_\alpha(x) \to \varrho_{\alpha,\omega} = c_+|x|^{\alpha+\mathrm{i}\omega} + c_-|x|^{\alpha-\mathrm{i}\omega}, \qquad \omega \geq 0\,.$$

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Summing over α, ω and imposing S to be real,

$$S = \int d\varrho(x)\,\mathcal{L}\,, \qquad d\varrho(x) = \prod_\mu \left[\sum_n g_n \sum_\omega d\varrho_{\alpha_n,\omega}(x^\mu)\right]$$

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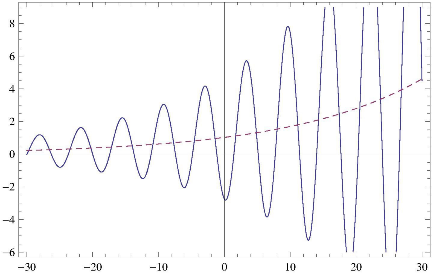
where

$$\varrho_{\alpha,\omega}(x) = \frac{x^\alpha}{\Gamma(\alpha+1)}\left[1 + A_{\alpha,\omega}\cos\left(\omega\ln\frac{|x|}{\ell_\infty}\right) + B_{\alpha,\omega}\sin\left(\omega\ln\frac{|x|}{\ell_\infty}\right)\right]$$

$A_{\alpha,\omega}$ and $B_{\alpha,\omega} \in \mathbb{R}$. Form of measure also dictated by fractal geometry arguments.

09/33– Example 3: Log-oscillating measure

Represents deterministic (multi)fractals (integrals on self-similar fractals can be approximated by fractional integrals with $\alpha \sim d_{\rm H}$ [Ren et al. 1996–2003; Nigmatullin & Le Méhauté 2003]).



10/33– Example 3: Discrete scale invariance

Oscillatory part of ϱ log-periodic under the transformation

$$\ln\frac{|x|}{\ell_\infty}\to\ln\frac{|x|}{\ell_\infty}+\frac{2\pi n}{\omega}\,,\qquad n=0,1,2,\dots$$

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DSIs appear in chaotic systems [Sornette 1998].

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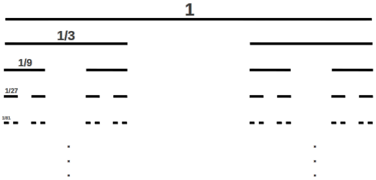
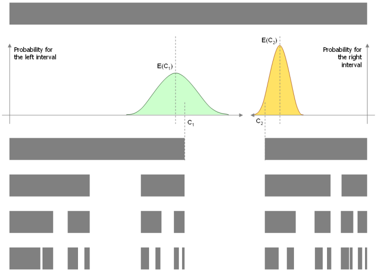
5. $d_S \leq d_H$. Depends on model.

12/33– Cantor set

Deterministic

Random





13/33– Example: scalar field

$$S=\int \mathrm{d}^D x\, v(x)\,\left[-\frac{1}{2}\phi\,\mathcal{K}\,\phi-V(\phi)\right]\,.$$

The symmetries of $\mathcal{L}$ determine the Laplace–Beltrami operator $\mathcal{K}$

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- q-Laplacian:
  $\mathcal{K} = \Box_q := \eta^{\mu\nu} \frac{\partial}{\partial q^\mu(x^\mu)} \frac{\partial}{\partial q^\nu(x^\nu)} = \eta^{\mu\nu} \frac{1}{v_\mu} \partial_\mu \left[\frac{1}{v_\nu}\partial_\nu\,\cdot\,\right] = \mathcal{K}^\dagger$. Not trivial (physical momenta conjugate to x, not q!). It is a fractal.

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- Multifractional Laplacian:
  $\mathcal{K}=\mathcal{K}_*:=\eta^{\mu\nu}\left[\frac{1}{2}(\partial_\mu^\gamma\partial_\nu^\gamma+{}_\infty\bar{\partial}_\mu^\gamma{}_\infty\bar{\partial}_\nu^\gamma)\,\cdot\,\right]=\mathcal{K}_*^\dagger$.

15/33– Scales hierarchy

- *Boundary-effect regime* ($\ell \sim \ell_\infty$). $|x|/\ell_\infty \sim 1$, $\varrho(x) \sim \ln|x|$, natural relation with κ-Minkowski noncommutative spacetimes ($\ell_\infty = \ell_{\rm Pl}$).
- *Oscillatory transient regime* ($\ell_\omega = \lambda_\omega \ell_\infty < \ell \ll \ell_*$). Notion of dim. and vol. ambiguous unless averaged. DSI.
- *Multifractional regime* ($\ell_\omega \ll \ell \lesssim \ell_*$). Mesoscopic scales, average of the measure:
  $\varrho_\alpha(x) := \langle \varrho_{\alpha,\omega}(x) \rangle \propto |x|^\alpha$, $\qquad d\varrho(x) \sim \sum_\alpha g_\alpha d\varrho_\alpha(x)$.
  Dimensional flow. **Continuous symmetries (affinities) emerge**.
- *Classical regime* ($\ell \gg \ell_*$). Ordinary Poincaré-invariant field theory on Minkowski spacetime recovered, $\varrho(x) \sim \varrho_1(x) = x$. Dim. of spacetime is $d_{\rm H} = d_{\rm S} = 4 - \epsilon$.

16/33– Status

	$\Box$, $\Box^\dagger$	$\mathcal{D}^2$	$\Box_q$	$\mathcal{K}_*$
Momentum transform	✗?	✓	✓	?
Relativistic mechanics	✓	✓	✓	?
Perturbative field theory	✓?	✓	✓?	✓?
Symmetries and dynamics of scalar (Q)FT	?	✓	✓	?
Scalar QFT propagator	?	✓	✓?	✓?
Electrodynamics	?	✓	✓	?
Perturbative renormalizability	?	✗	✗	✓?
Avoids Collins *et al.*	?	✓	✓	?
Phenomenology (obs. constraints)	?	✓	?	?
Gravity	✓	✓	✓	?
Cosmology	✓?	✓	✓	?

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- Only theory with q derivatives: What expected in a 'covariant' description of a fractal: nontrivial geometric and differential structure at all points. Vielbeins move the measure 'hole' around and maintain the anomalous scaling properties.

18/33– Action: Will o' the WIST

$$
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\begin{aligned}
S \;&:=\; \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, v\,\sqrt{-g}\,\left[\mathcal{R} - \omega\mathcal{D}_\mu v\mathcal{D}_\nu v - U(v)\right] + S_\mathrm{m}\\
&=\; \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, \mathrm{e}^{\frac{1}{\beta}\Phi}\sqrt{-g}\,\left(\mathcal{R} - \frac{9\omega}{4\beta^2}\mathrm{e}^{\frac{2}{\beta}\Phi}\partial_\mu\Phi\partial^\mu\Phi - U\right) + S_\mathrm{m}\,.
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19/33– $D = 4$ Einstein and Friedmann equations

Einstein frame: $\bar{g}_{\mu\nu} = \mathrm{e}^{\Phi} g_{\mu\nu}$. $\Omega = (9\omega/4)\mathrm{e}^{2\Phi} - 3/2$.

$$\kappa^2 \bar{T}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}(\bar{R} - \mathrm{e}^{-\Phi} U) - \Omega\left(\partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2}\bar{g}_{\mu\nu}\partial_\sigma \Phi \bar{\partial}^\sigma \Phi\right)$$

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$$\kappa^2 \bar{T}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}(\bar{R} - \mathrm{e}^{-\Phi}U) - \Omega\left(\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}\bar{g}_{\mu\nu}\partial_\sigma\Phi\bar{\partial}^\sigma\Phi\right)$$

Cosmology:

$$H^2 = \frac{\kappa^2}{3}\bar{\rho} + \frac{\Omega}{2}\frac{\dot{v}^2}{v^2} + \frac{U(v)}{6v} - \frac{\mathrm{K}}{a^2}\,,$$

$$2\dot{H} - \frac{2\mathrm{K}}{a^2} + \kappa^2(\bar{\rho} + \bar{P}) = -\Omega\frac{\dot{v}^2}{v^2}\,.$$

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Profile $v(x)$ fixed *a priori*, 'potential' $U(v)$ *reconstructed*.

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$$a(t) = \left(1 + \sqrt{\frac{t_*}{t}}\right)^{\frac{3}{8}} \exp\left\{\frac{9}{8}\left[H_0\frac{t}{t_*} + \sqrt{\frac{t}{t_*}} - \frac{t}{t_*}\ln\left(1 + \sqrt{\frac{t_*}{t}}\right)\right]\right\}.$$

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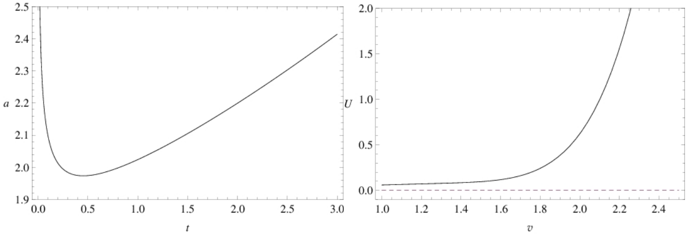
Minimum of the 'potential' U at late times
$U_{\rm min} = U(v=1) = 6H_0^2$.

19/33– Flat vacuum solution ($\rho = P = \mathsf{K} = 0 = \omega$)



21/33– Cosmological constant

$U(v)$ determined for self-consistency by the dynamics, effective cosmological constant:

$$\Lambda(x, \ell_n) \equiv \frac{1}{2\kappa^2} U[v(x, \ell_n)].$$

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Energy scale of dynamical Λ determined by the structure of the measure. At late times, $U(v \sim 1) \sim H_0^2$.
Rigidity of the multi-scale *Ansatz* allows to constrain the scales ℓ_n against various experiments and test the prediction for the whole evolution of the universe. Easily falsifiable models.

22/33– Frames and fractals

- Inertial frames clearly interpreted: Multiscale frames map a curvilinear coordinate system to the Cartesian one *with the same measure structure*.

$$e_\mu^{\ J} := \frac{v(x^J)}{v(x'^\mu)}\bar{e}_\mu^{\ J} = \frac{v(x^J)}{v(x'^\mu)}\frac{\partial x^J}{\partial x'^\mu} = \frac{\partial q^J}{\partial q'^\mu}\,.$$

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- Line element and metric:

$$\mathrm{d}s^2 := g_{\mu\nu}\,\mathrm{d}q^\mu \otimes \mathrm{d}q^\nu\,, \qquad g_{\mu\nu} := \eta_{IJ}e_\mu^{\ I}e_\nu^{\ J} \not\propto \bar{g}_{\mu\nu}\,.$$

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- Gravity and cosmology easy to work out via $x \to q(x)$ mapping.

23/33– Action and Einstein equations

$$ {}^q\Gamma^\rho_{\mu\nu} := \tfrac{1}{2} g^{\rho\sigma}\left(\frac{1}{v_\mu}\partial_\mu g_{\nu\sigma} + \frac{1}{v_\nu}\partial_\nu g_{\mu\sigma} - \frac{1}{v_\sigma}\partial_\sigma g_{\mu\nu}\right)\,, $$

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Action:

$$ S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, v\,\sqrt{-g}\,({}^q R - 2\Lambda) + S_\mathrm{m}\,. $$

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Einstein equations:

$$
{}^qR_{\mu\nu} - \frac{1}{2} g_{\mu\nu}({}^qR - 2\Lambda) = \kappa^2\,{}^qT_{\mu\nu}\,.
$$

24/33– Cosmology

$$\frac{H^2}{v^2} = \frac{\kappa^2}{3}\rho + \frac{\Lambda}{3} - \frac{\textsf{K}}{a^2}\,,$$
$$\dot\rho + 3H(\rho + P) = 0\,.$$

Ordinary slow-roll approximation unnecessary.

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Ordinary slow-roll approximation unnecessary.
Geometric time coordinate

$$q(t) = \int^t \mathrm{d}t'\, v(t')\,.$$

25/33– ‘Power-law’ solutions

$$\rho = \rho_0\, a^{-\frac{2}{p}}\,, \qquad a(t) = \left[\frac{q(t)}{t_*}\right]^p\,, \qquad p := \frac{2}{(D-1)(1+w)}\,,$$

Hubble parameter

$$H = p\frac{\dot q(t)}{q(t)} = p\frac{v(t)}{q(t)}\,.$$

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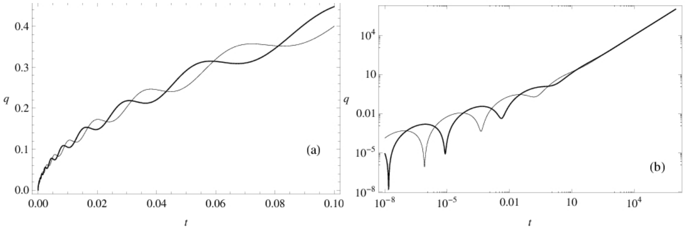
From now on choose log-oscillating measure

$$\begin{aligned} q(t) &= t + \frac{t_*}{\alpha}\left(\frac{t}{t_*}\right)^\alpha F_\omega(\ln t)\,, \\ F_\omega(\ln t) &= 1 + A\cos\left[\omega \ln\left(\frac{t}{t_{\rm Pl}}\right)\right] + B\sin\left[\omega \ln\left(\frac{t}{t_{\rm Pl}}\right)\right]. \end{aligned}$$

26/33– Geometric coordinate



$H = 0$ at peaks and troughs, log-oscillations end after some time.

27/33– e-folds and cycles

Fully analytic properties.

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Slope of an expanding/contracting phase:

$$\delta_\uparrow \approx p\left(\alpha+\frac{\omega}{\pi}\ln\frac{1+A}{1-A}\right)>p\,,\qquad \delta_\downarrow \approx p\left(\alpha-\frac{\omega}{\pi}\ln\frac{1+A}{1-A}\right).$$

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Average slope of the trend of minima (or maxima):

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Net number of e-foldings per cycle:

$$\mathcal{N}_{\uparrow\downarrow} := \ln\frac{a_{m+1}}{a_m} = (\delta_\uparrow + \delta_\downarrow)\frac{\pi}{\omega} \approx \frac{2\pi\alpha p}{\omega}\,.$$

28/33– Big bang

- Big bang removed by a homogeneous contribution to the measure (integration constant):

$$q(t) \to t_{\rm bb} + q(t)\,.$$

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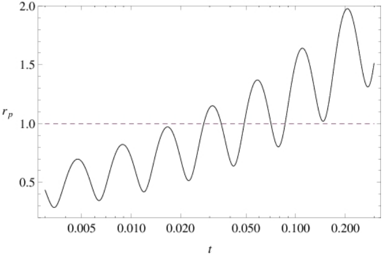
• Alternative: $t_{\rm bb}=0=\alpha$ (oscillations with constant amplitude).

29/33– Alternative to inflation? $(0 < p < 1)$

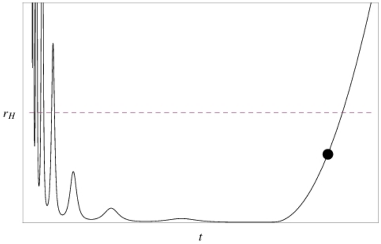
Horizon problem solved without invoking acceleration-inducing matter: particle horizon shrinks because of geometry!



Flatness problem not solved, unfortunately.

30/33– Cyclic mild inflation ($p \gtrsim 1$)

Horizon and flatness problem solved if mildly inflating matter is added.



31/33– Inflationary spectra

$$\tilde{k} = k_* \left[\frac{k_*}{k} + \text{sgn}(k)\frac{k_*}{\sqrt{3}\alpha}\left|\frac{k_*}{k}\right|^\alpha F_\omega(\ln|k/\sqrt{3}|)\right]^{-1} .$$

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General behaviour:

$$P_{\rm s,t} = \mathcal{A}_{\rm s,t} \tilde{k}^{n-1} \sim \mathcal{A}_{\rm s,t} \left(\frac{k}{k_*} \right)^{\alpha(n-1)} [F_\omega(\ln k)]^{1-n} .$$

Scale invariance without slow-roll approximation and log-oscillating pattern!!

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Scale invariance without slow-roll approximation and log-oscillating pattern!! Power-law inflation ($\nu_p = 1/(p-1)$):

$$\mathcal{P}_{\mathrm{s}} = \frac{\kappa^2}{2} (p-1)^2 p \left[2^{\nu_p} \frac{\Gamma\left(\frac{3}{2} + \nu_p\right)}{\Gamma\left(\frac{3}{2}\right)} \right]^2 \frac{1}{(2\pi q)^2} ,$$

$$\mathcal{P}_{\mathrm{t}} = 8\kappa^2 (\cdots)(2\pi q)^{-2} .$$

32/33– Observables

Tensor-to-scalar ratio:

$$r := \frac{\mathcal{P}_{\rm t}}{\mathcal{P}_{\rm s}} = \frac{16}{p}\,, \qquad p \gtrsim 80\,.$$

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Scalar (or tensor) spectral index:

$$n-1 = -2\frac{\mathrm{d}\ln\tilde{k}}{\mathrm{d}\ln k}\frac{1}{p-1-p\dot{v}/(vH)} \sim \alpha(n-1)\,.$$

33/33– Discussion

- Assuming the integro-differential structure of spacetime behaves as a multifractal leads to novel scenarios in particle physics and cosmology.
- Analytic cosmological solutions to be studies (stability, cosmic evolution with matter and radiation, etc.).
- Intriguing features purely generated by geometry (big bounces, acceleration without inflaton, alternative to inflation, mild inflation, cyclic cosmology).
- Power spectra at hand (with S. Tsujikawa, in progress).

33/33– Discussion

どうもありがとうございました！

¡Muchas gracias!

Thank you!

Danke schön!