Towards solving generic singularity problem

Włodzimierz Piechocki

Department of Fundamental Research
National Centre for Nuclear Research
Warsaw, Poland

Quantum Cosmology Group:
Ewa Czuchry, Orest Hrycyna, Przemek Małkiewicz, Kuba Mielczarek, and WP

http://quantcosmo.ncbj.pl/

OUTLINE

Introduction

Evidence for the existence of the cosmological singularity

- observational cosmology:
  the Universe has been expanding for almost 14 billion years (emerged from a state with extremely high energy densities of physical fields)
- theoretical cosmology:
  almost all known general relativity models of the Universe (Lemaître, Kasner, Friedmann, Bianchi, Szekeres, ..., BKL) predict the existence of cosmological singularities (diverging gravitational and matter field invariants, incomplete geodesics).

Existence of the cosmological singularities in solutions to GR may mean that this classical theory is incomplete.

Expectation: quantization may heal the cosmological singularity.

Introduction

Evidence for the existence of the cosmological singularity

- observational cosmology:
  the Universe has been expanding for almost 14 billion years (emerged from a state with extremely high energy densities of physical fields)
- theoretical cosmology:
  almost all known general relativity models of the Universe (Lemaître, Kasner, Friedmann, Bianchi, Szekeres, ..., BKL) predict the existence of cosmological singularities (diverging gravitational and matter field invariants, incomplete geodesics).

Existence of the cosmological singularities in solutions to GR may mean that this classical theory is incomplete.

Expectation: quantization may heal the cosmological singularity.

Introduction

Evidence for the existence of the cosmological singularity

- observational cosmology:
  the Universe has been expanding for almost 14 billion years (emerged from a state with extremely high energy densities of physical fields)
- theoretical cosmology:
  almost all known general relativity models of the Universe (Lemaître, Kasner, Friedmann, Bianchi, Szekeres, ..., BKL) predict the existence of cosmological singularities (diverging gravitational and matter field invariants, incomplete geodesics).

Existence of the cosmological singularities in solutions to GR may mean that this classical theory is incomplete.

Expectation: quantization may heal the cosmological singularity.

Introduction

Evidence for the existence of the cosmological singularity

- observational cosmology:
  the Universe has been expanding for almost 14 billion years (emerged from a state with extremely high energy densities of physical fields)
- theoretical cosmology:
  almost all known general relativity models of the Universe (Lemaître, Kasner, Friedmann, Bianchi, Szekeres, ..., BKL) predict the existence of cosmological singularities (diverging gravitational and matter field invariants, incomplete geodesics).

Existence of the cosmological singularities in solutions to GR may mean that this classical theory is incomplete.

Expectation: quantization may heal the cosmological singularity.

Introduction

Evidence for the existence of the cosmological singularity

- observational cosmology:
  the Universe has been expanding for almost 14 billion years (emerged from a state with extremely high energy densities of physical fields)
- theoretical cosmology:
  almost all known general relativity models of the Universe (Lemaître, Kasner, Friedmann, Bianchi, Szekeres, ..., BKL) predict the existence of cosmological singularities (diverging gravitational and matter field invariants, incomplete geodesics).

Existence of the cosmological singularities in solutions to GR may mean that this classical theory is incomplete.

Expectation: quantization may heal the cosmological singularity.

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition:
  quantum phase ⇄ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition:
  quantum phase ⇄ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

Some intriguing questions concerning the quantum phase of the Universe:

- What is the energy scale?
- What is the mechanism of the transition: quantum phase $\rightleftarrows$ classical phase?
- How to relate theory with cosmic observations?
  - What is the origin of inflation?
  - What is the origin of tiny fluctuations visible in CMB?
  - What is the origin of primordial gravitational waves?
  - What is the origin of visible matter?
  - What is the origin and nature of dark matter?
  - What is the origin and nature of dark energy?
- If the notion of time is well defined:
  - How long had the quantum phase lasted?
  - What was before the quantum phase?

Introduction (cont)

- Canonical quantization based on the Holst action and loop geometry
  - Dirac's LQC[1] := 'first quantize then impose constraints'
  - RPS LQC[2] := 'first solve constraints then quantize'
- Coherent states[3] and canonical[4] quantizations based on the Hilbert-Einstein action

[1] Ashtekar, Bojowald, Lewandowski, Rovelli, Thiemann, ...
[2] P. Dzierżak, P. Małkiewicz, J. Mielczarek, WP, ...
[3] H. Bergeron, J.P. Gazeau, P. Małkiewicz, W.P.,...
[4] D.M. Chitre, C.W. Misner, M.P. Ryann, ...

Introduction (cont)

- Canonical quantization based on the Holst action and loop geometry
  - Dirac's LQC[1] := 'first quantize then impose constraints'
  - RPS LQC[2] := 'first solve constraints then quantize'
- Coherent states[3] and canonical[4] quantizations based on the Hilbert-Einstein action

[1] Ashtekar, Bojowald, Lewandowski, Rovelli, Thiemann, . . .
[2] P. Dzierżak, P. Małkiewicz, J. Mielczarek, WP, . . .
[3] H. Bergeron, J.P. Gazeau, P. Małkiewicz, W.P.,. . .
[4] D.M. Chitre, C.W. Misner, M.P. Ryann, . . .

Introduction (cont)

- Canonical quantization based on the Holst action and loop geometry
  - Dirac's LQC[1] := 'first quantize then impose constraints'
  - RPS LQC[2] := 'first solve constraints then quantize'
- Coherent states[3] and canonical[4] quantizations based on the Hilbert-Einstein action

[1] Ashtekar, Bojowald, Lewandowski, Rovelli, Thiemann, ...
[2] P. Dzierżak, P. Małkiewicz, J. Mielczarek, WP, ...
[3] H. Bergeron, J.P. Gazeau, P. Małkiewicz, W.P.,...
[4] D.M. Chitre, C.W. Misner, M.P. Ryann, ...

Introduction (cont)

- Canonical quantization based on the Holst action and loop geometry
  - Dirac's LQC[1] $:=$ 'first quantize then impose constraints'
  - RPS LQC[2] $:=$ 'first solve constraints then quantize'
- Coherent states[3] and canonical[4] quantizations based on the Hilbert-Einstein action

[1] Ashtekar, Bojowald, Lewandowski, Rovelli, Thiemann, ...
[2] P. Dzierżak, P. Małkiewicz, J. Mielczarek, WP, ...
[3] H. Bergeron, J.P. Gazeau, P. Małkiewicz, W.P.,...
[4] D.M. Chitre, C.W. Misner, M.P. Ryann, ...

Quantum FRW model: summary of results obtained within RPS LQC approach

- Cosmic singularity problem of FRW model can be resolved by using the loop geometry: big bang turns into big bounce
- Discreteness of the spectra of the volume operator may favor a foamy structure of space at short distances: no dispersion of cosmic photons[5] up to the energy 5×10^{17} GeV
- Evolution of quantum phase can be described in terms of self-adjoint true (physical) Hamiltonian
  - expectation values of quantum variables coincide with corresponding classical variables
  - Heisenberg's uncertainty relation is perfectly satisfied during the entire evolution of the universe.

[5] F. Aharonian *et al.*, Phys. Rev. Lett. **101**, 170402 (2008), 'Limits on an Energy Dependence of the Speed of Light from a Flare ...'

Quantum FRW model: summary of results obtained within RPS LQC approach

- Cosmic singularity problem of FRW model can be resolved by using the loop geometry: big bang turns into big bounce
- Discreteness of the spectra of the volume operator may favor a foamy structure of space at short distances: no dispersion of cosmic photons[5] up to the energy 5×10^{17} GeV
- Evolution of quantum phase can be described in terms of self-adjoint true (physical) Hamiltonian
  - expectation values of quantum variables coincide with corresponding classical variables
  - Heisenberg's uncertainty relation is perfectly satisfied during the entire evolution of the universe.

[5] F. Aharonian *et al.*, Phys. Rev. Lett. **101**, 170402 (2008), 'Limits on an Energy Dependence of the Speed of Light from a Flare ...'

Quantum FRW model: summary of results obtained within RPS LQC approach

- Cosmic singularity problem of FRW model can be resolved by using the loop geometry: big bang turns into big bounce
- Discreteness of the spectra of the volume operator may favor a foamy structure of space at short distances: no dispersion of cosmic photons[5] up to the energy 5×10^{17} GeV
- Evolution of quantum phase can be described in terms of self-adjoint true (physical) Hamiltonian
  - expectation values of quantum variables coincide with corresponding classical variables
  - Heisenberg's uncertainty relation is perfectly satisfied during the entire evolution of the universe.

[5] F. Aharonian *et al.*, Phys. Rev. Lett. **101**, 170402 (2008), 'Limits on an Energy Dependence of the Speed of Light from a Flare ...'

Quantum FRW model: summary of results obtained within RPS LQC approach

- Cosmic singularity problem of FRW model can be resolved by using the loop geometry: big bang turns into big bounce
- Discreteness of the spectra of the volume operator may favor a foamy structure of space at short distances: no dispersion of cosmic photons[5] up to the energy 5×10^{17} GeV
- Evolution of quantum phase can be described in terms of self-adjoint true (physical) Hamiltonian
  - expectation values of quantum variables coincide with corresponding classical variables
  - Heisenberg's uncertainty relation is perfectly satisfied during the entire evolution of the universe.

[5] F. Aharonian *et al.*, Phys. Rev. Lett. **101**, 170402 (2008), 'Limits on an Energy Dependence of the Speed of Light from a Flare ...'

Quantum FRW model: summary of results obtained within RPS LQC approach

- Cosmic singularity problem of FRW model can be resolved by using the loop geometry: big bang turns into big bounce
- Discreteness of the spectra of the volume operator may favor a foamy structure of space at short distances: no dispersion of cosmic photons[5] up to the energy 5×10^{17} GeV
- Evolution of quantum phase can be described in terms of self-adjoint true (physical) Hamiltonian
  - expectation values of quantum variables coincide with corresponding classical variables
  - Heisenberg's uncertainty relation is perfectly satisfied during the entire evolution of the universe.

[5] F. Aharonian *et al.*, Phys. Rev. Lett. **101**, 170402 (2008), 'Limits on an Energy Dependence of the Speed of Light from a Flare ...'

Quantum FRW model: summary of results obtained within RPS LQC approach

- Cosmic singularity problem of FRW model can be resolved by using the loop geometry: big bang turns into big bounce
- Discreteness of the spectra of the volume operator may favor a foamy structure of space at short distances: no dispersion of cosmic photons[5] up to the energy 5×10^{17} GeV
- Evolution of quantum phase can be described in terms of self-adjoint true (physical) Hamiltonian
  - expectation values of quantum variables coincide with corresponding classical variables
  - Heisenberg's uncertainty relation is perfectly satisfied during the entire evolution of the universe.

[5] F. Aharonian *et al.*, Phys. Rev. Lett. **101**, 170402 (2008), 'Limits on an Energy Dependence of the Speed of Light from a Flare ...'

Main papers:

[1] P. Dzierżak, P. Małkiewicz and W. P.,
'Turning big bang into big bounce: I. Classical dynamics',
Phys. Rev. D **80**, 104001 (2009).
[2] P. Małkiewicz and W. P.,
'Turning big bang into big bounce: II. Quantum dynamics'.
Class. Quant. Grav. **27** (2010) 225018.
[3] P. Małkiewicz and W. P.,
'Energy scale of the Big Bounce',
Phys. Rev. D **80**, 063506 (2009).
[4] J. Mielczarek and W. P.,
'Observables for FRW model with cosmological constant'.
Phys. Rev. D **82** (2010) 043529.
[5] J. Mielczarek and W. P.,
'Evolution in bouncing quantum cosmology',
Class. Quant. Grav. **29** (2012) 065022.
[6] J. Mielczarek and W. P.,
'Gaussian state for the bouncing quantum cosmology ',
Phys. Rev. D **86** (2012) 083508.
[7] J-P. Gazeau, J. Mielczarek and W. P.,
'Quantum states of the bouncing universe',
Phys. Rev. D **87**, 123508 (2013).

Challenge in Cosmology:

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)
- Bianchi type metric is dynamically unstable in the evolution towards the singularity (breaking of homogeneity)
- BKL scenario is thought to be generic solution to GR near CS
  - does not rely on any symmetry conditions;
  - corresponds to non-zero measure subset of all initial conditions;
  - solution is stable against perturbation of initial conditions
- BKL appears in the low energy limit of superstring models
- application of non-singular quantum BKL theory
  - realistic model of the very early Universe
  - may help in the construction of the theory unifying gravitation and quantum physics.

Challenge in Cosmology:

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)
- Bianchi type metric is dynamically unstable in the evolution towards the singularity (breaking of homogeneity)
- BKL scenario is thought to be generic solution to GR near CS
  - does not rely on any symmetry conditions;
  - corresponds to non-zero measure subset of all initial conditions;
  - solution is stable against perturbation of initial conditions
- BKL appears in the low energy limit of superstring models
- application of non-singular quantum BKL theory
  - realistic model of the very early Universe
  - may help in the construction of the theory unifying gravitation and quantum physics.

Challenge in Cosmology:

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)
- Bianchi type metric is dynamically unstable in the evolution towards the singularity (breaking of homogeneity)
- BKL scenario is thought to be generic solution to GR near CS
  - does not rely on any symmetry conditions;
  - corresponds to non-zero measure subset of all initial conditions;
  - solution is stable against perturbation of initial conditions
- BKL appears in the low energy limit of superstring models
- application of non-singular quantum BKL theory
  - realistic model of the very early Universe
  - may help in the construction of the theory unifying gravitation and quantum physics.

Challenge in Cosmology:

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)
- Bianchi type metric is dynamically unstable in the evolution towards the singularity (breaking of homogeneity)
- BKL scenario is thought to be generic solution to GR near CS
  - does not rely on any symmetry conditions;
  - corresponds to non-zero measure subset of all initial conditions;
  - solution is stable against perturbation of initial conditions
- BKL appears in the low energy limit of superstring models
- application of non-singular quantum BKL theory
  - realistic model of the very early Universe
  - may help in the construction of the theory unifying gravitation and quantum physics.

Challenge in Cosmology:

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)
- Bianchi type metric is dynamically unstable in the evolution towards the singularity (breaking of homogeneity)
- BKL scenario is thought to be generic solution to GR near CS
  - does not rely on any symmetry conditions;
  - corresponds to non-zero measure subset of all initial conditions;
  - solution is stable against perturbation of initial conditions
- BKL appears in the low energy limit of superstring models
- application of non-singular quantum BKL theory
  - realistic model of the very early Universe
  - may help in the construction of the theory unifying gravitation and quantum physics.

Challenge in Cosmology:

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)
- Bianchi type metric is dynamically unstable in the evolution towards the singularity (breaking of homogeneity)
- BKL scenario is thought to be generic solution to GR near CS
  - does not rely on any symmetry conditions;
  - corresponds to non-zero measure subset of all initial conditions;
  - solution is stable against perturbation of initial conditions
- BKL appears in the low energy limit of superstring models
- application of non-singular quantum BKL theory
  - realistic model of the very early Universe
  - may help in the construction of the theory unifying gravitation and quantum physics.

Challenge in Cosmology:

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)
- Bianchi type metric is dynamically unstable in the evolution towards the singularity (breaking of homogeneity)
- BKL scenario is thought to be generic solution to GR near CS
  - does not rely on any symmetry conditions;
  - corresponds to non-zero measure subset of all initial conditions;
  - solution is stable against perturbation of initial conditions
- BKL appears in the low energy limit of superstring models
- application of non-singular quantum BKL theory
  - realistic model of the very early Universe
  - may help in the construction of the theory unifying gravitation and quantum physics.

Challenge in Cosmology:

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)
- Bianchi type metric is dynamically unstable in the evolution towards the singularity (breaking of homogeneity)
- BKL scenario is thought to be generic solution to GR near CS
  - does not rely on any symmetry conditions;
  - corresponds to non-zero measure subset of all initial conditions;
  - solution is stable against perturbation of initial conditions
- BKL appears in the low energy limit of superstring models
- application of non-singular quantum BKL theory
  - realistic model of the very early Universe
  - may help in the construction of the theory unifying gravitation and quantum physics.

Challenge in Cosmology:

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)
- Bianchi type metric is dynamically unstable in the evolution towards the singularity (breaking of homogeneity)
- BKL scenario is thought to be generic solution to GR near CS
  - does not rely on any symmetry conditions;
  - corresponds to non-zero measure subset of all initial conditions;
  - solution is stable against perturbation of initial conditions
- BKL appears in the low energy limit of superstring models
- application of non-singular quantum BKL theory
  - realistic model of the very early Universe
  - may help in the construction of the theory unifying gravitation and quantum physics.

Challenge in Cosmology:

Quantization of the Belinskii-Khalatnikov-Lifshitz scenario (1963-82).

- FRW metric is dynamically unstable in the evolution towards the singularity (breaking of isotropy)
- Bianchi type metric is dynamically unstable in the evolution towards the singularity (breaking of homogeneity)
- BKL scenario is thought to be generic solution to GR near CS
  - does not rely on any symmetry conditions;
  - corresponds to non-zero measure subset of all initial conditions;
  - solution is stable against perturbation of initial conditions
- BKL appears in the low energy limit of superstring models
- application of non-singular quantum BKL theory
  - realistic model of the very early Universe
  - may help in the construction of the theory unifying gravitation and quantum physics.

The Bianchi IX model:

- Dynamics of Bianchi-IX is the best prototype for the BKL scenario[6]
- Questions to answer:
  - What happens to the oscillatory/chaotic dynamics after the imposition of quantum rules onto the dynamics?
  - What happens to the classical singularity of the Bianchi-IX at the quantum level?
  - What is the quantum generation of primordial GW for the Bianchi IX model?
- Successful quantization of the Bianchi IX model would open the door to the quantization of the BKL scenario.

[6] V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, "Oscillatory approach to a singular point in the relativistic cosmology", Adv. Phys. **19** (1970) 525.

The Bianchi IX model:

- Dynamics of Bianchi-IX is the best prototype for the BKL scenario[6]
- Questions to answer:
  - What happens to the oscillatory/chaotic dynamics after the imposition of quantum rules onto the dynamics?
  - What happens to the classical singularity of the Bianchi-IX at the quantum level?
  - What is the quantum generation of primordial GW for the Bianchi IX model?
- Successful quantization of the Bianchi IX model would open the door to the quantization of the BKL scenario.

[6] V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, "Oscillatory approach to a singular point in the relativistic cosmology", Adv. Phys. **19** (1970) 525.

The Bianchi IX model:

- Dynamics of Bianchi-IX is the best prototype for the BKL scenario[6]
- Questions to answer:
  - What happens to the oscillatory/chaotic dynamics after the imposition of quantum rules onto the dynamics?
  - What happens to the classical singularity of the Bianchi-IX at the quantum level?
  - What is the quantum generation of primordial GW for the Bianchi IX model?
- Successful quantization of the Bianchi IX model would open the door to the quantization of the BKL scenario.

[6] V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, "Oscillatory approach to a singular point in the relativistic cosmology", Adv. Phys. **19** (1970) 525.

The Bianchi IX model:

- Dynamics of Bianchi-IX is the best prototype for the BKL scenario[6]
- Questions to answer:
  - What happens to the oscillatory/chaotic dynamics after the imposition of quantum rules onto the dynamics?
  - What happens to the classical singularity of the Bianchi-IX at the quantum level?
  - What is the quantum generation of primordial GW for the Bianchi IX model?
- Successful quantization of the Bianchi IX model would open the door to the quantization of the BKL scenario.

[6] V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, “Oscillatory approach to a singular point in the relativistic cosmology”, Adv. Phys. **19** (1970) 525.

The Bianchi IX model:

- Dynamics of Bianchi-IX is the best prototype for the BKL scenario[6]
- Questions to answer:
  - What happens to the oscillatory/chaotic dynamics after the imposition of quantum rules onto the dynamics?
  - What happens to the classical singularity of the Bianchi-IX at the quantum level?
  - What is the quantum generation of primordial GW for the Bianchi IX model?
- Successful quantization of the Bianchi IX model would open the door to the quantization of the BKL scenario.

[6]V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, “Oscillatory approach to a singular point in the relativistic cosmology”, Adv. Phys. **19** (1970) 525.

The Bianchi IX model:

- Dynamics of Bianchi-IX is the best prototype for the BKL scenario[6]
- Questions to answer:
  - What happens to the oscillatory/chaotic dynamics after the imposition of quantum rules onto the dynamics?
  - What happens to the classical singularity of the Bianchi-IX at the quantum level?
  - What is the quantum generation of primordial GW for the Bianchi IX model?
- Successful quantization of the Bianchi IX model would open the door to the quantization of the BKL scenario.

[6] V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, "Oscillatory approach to a singular point in the relativistic cosmology", Adv. Phys. **19** (1970) 525.

Metric of the Bianchi IX model

The general form of a line element of the non-diagonal Bianchi IX model, in the synchronous reference system, reads:

$$ds^2 = dt^2 - \gamma_{ab}(t) e^a_\alpha e^b_\beta dx^\alpha dx^\beta, \qquad (1)$$

where $a, b, \dots$ run from 1 to 3 and label frame vectors; $\alpha, \beta, \dots$ take values $1, 2, 3$ and concern space coordinates, and where γ_{ab} is a spatial metric.

The homogeneity of the Bianchi IX model means that the three independent differential 1-forms $e^a_\alpha dx^\alpha$ are invariant under the transformations of the isometry group of the Bianchi IX model.

The cosmological time variable t is redefined as follows:

$$dt = \sqrt{\gamma}\, d\tau, \qquad \gamma := det[\gamma_{ab}] \qquad (2)$$

where γ is the volume density, and $\gamma \to 0$ denotes the singularity.

Metric of the Bianchi IX model

The general form of a line element of the non-diagonal Bianchi IX model, in the synchronous reference system, reads:

$$ds^2 = dt^2 - \gamma_{ab}(t) e^a_\alpha e^b_\beta dx^\alpha dx^\beta, \tag{1}$$

where $a, b, \ldots$ run from 1 to 3 and label frame vectors; $\alpha, \beta, \ldots$ take values $1, 2, 3$ and concern space coordinates, and where γ_{ab} is a spatial metric.

The homogeneity of the Bianchi IX model means that the three independent differential 1-forms $e^a_\alpha dx^\alpha$ are invariant under the transformations of the isometry group of the Bianchi IX model.

The cosmological time variable t is redefined as follows:

$$dt = \sqrt{\gamma}\, d\tau, \qquad \gamma := det[\gamma_{ab}] \tag{2}$$

where γ is the volume density, and $\gamma \to 0$ denotes the singularity.

Metric of the Bianchi IX model

The general form of a line element of the non-diagonal Bianchi IX model, in the synchronous reference system, reads:

$$ds^2 = dt^2 - \gamma_{ab}(t) e^a_\alpha e^b_\beta dx^\alpha dx^\beta, \tag{1}$$

where $a, b, \ldots$ run from 1 to 3 and label frame vectors; $\alpha, \beta, \ldots$ take values $1, 2, 3$ and concern space coordinates, and where γ_{ab} is a spatial metric.

The homogeneity of the Bianchi IX model means that the three independent differential 1-forms $e^a_\alpha dx^\alpha$ are invariant under the transformations of the isometry group of the Bianchi IX model.

The cosmological time variable t is redefined as follows:

$$dt = \sqrt{\gamma}\, d\tau, \qquad \gamma := det[\gamma_{ab}] \tag{2}$$

where γ is the volume density, and $\gamma \to 0$ denotes the singularity.

Equations of motion

Near the cosmological singularity one can assume[7]:

1. the stress-energy tensor components may be ignored
2. the Ricci tensor components R_a^0 have negligible influence on the dynamics
3. the anisotropy of space may grow without bound
4. rotations of the Kasner axes can be ignored, but oscillations are alive

which lead to the simplification of the mathematical form of the dynamics.

[7] V. A. Belinskii, I. M. Khalatnikov and M. P. Ryan, "The oscillatory regime near the singularity in Bianchi-type IX universes", Preprint order **469** (1971), Landau Institute for Theoretical Physics, Moscow (unpublished); published as sections 1 and 2 in: M. P. Ryan, Ann. Phys. **70** (1971) 301.

Equations of motion

Near the cosmological singularity one can assume[7]:

1. the stress-energy tensor components may be ignored
2. the Ricci tensor components R_a^0 have negligible influence on the dynamics
3. the anisotropy of space may grow without bound
4. rotations of the Kasner axes can be ignored, but oscillations are alive

which lead to the simplification of the mathematical form of the dynamics.

[7] V. A. Belinskii, I. M. Khalatnikov and M. P. Ryan, "The oscillatory regime near the singularity in Bianchi-type IX universes", Preprint order **469** (1971), Landau Institute for Theoretical Physics, Moscow (unpublished); published as sections 1 and 2 in: M. P. Ryan, Ann. Phys. **70** (1971) 301.

Equations of motion

Near the cosmological singularity one can assume[7]:

1. the stress-energy tensor components may be ignored
2. the Ricci tensor components R^0_a have negligible influence on the dynamics
3. the anisotropy of space may grow without bound
4. rotations of the Kasner axes can be ignored, but oscillations are alive

which lead to the simplification of the mathematical form of the dynamics.

[7] V. A. Belinskii, I. M. Khalatnikov and M. P. Ryan, “The oscillatory regime near the singularity in Bianchi-type IX universes”, Preprint order **469** (1971), Landau Institute for Theoretical Physics, Moscow (unpublished); published as sections 1 and 2 in: M. P. Ryan, Ann. Phys. **70** (1971) 301.

Equations of motion (cont)

Finally, the asymptotic form (near the cosmological singularity) of the dynamical equations of the non-diagonal Bianchi IX model reads:

$$\frac{d^2 \ln a}{d\tau^2} = \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{d\tau^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{d^2 \ln c}{d\tau^2} = a^2 - \frac{c}{b}, \qquad (3)$$

where $a = a(\tau),\ b = b(\tau),\ c = c(\tau)$ are scale factors.
The solutions to (3) must satisfy the condition:

$$\frac{d \ln a}{d\tau}\frac{d \ln b}{d\tau} + \frac{d \ln a}{d\tau}\frac{d \ln c}{d\tau} + \frac{d \ln b}{d\tau}\frac{d \ln c}{d\tau} = a^2 + \frac{b}{a} + \frac{c}{b}. \qquad (4)$$

Eq (3) can be obtained from the Lagrangian equations of motion with L in the form:

$$L := \dot{x}_1\dot{x}_2 + \dot{x}_1\dot{x}_3 + \dot{x}_2\dot{x}_3 + \exp(2x_1) + \exp(x_2 - x_1) + \exp(x_3 - x_2). \qquad (5)$$

Equations of motion (cont)

Finally, the asymptotic form (near the cosmological singularity) of the dynamical equations of the non-diagonal Bianchi IX model reads:

$$\frac{d^2 \ln a}{d\tau^2} = \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{d\tau^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{d^2 \ln c}{d\tau^2} = a^2 - \frac{c}{b}, \qquad (3)$$

where $a = a(\tau),\ b = b(\tau),\ c = c(\tau)$ are scale factors.
The solutions to (3) must satisfy the condition:

$$\frac{d \ln a}{d\tau}\frac{d \ln b}{d\tau} + \frac{d \ln a}{d\tau}\frac{d \ln c}{d\tau} + \frac{d \ln b}{d\tau}\frac{d \ln c}{d\tau} = a^2 + \frac{b}{a} + \frac{c}{b}. \qquad (4)$$

Eq (3) can be obtained from the Lagrangian equations of motion with L in the form:

$$L := \dot{x}_1\dot{x}_2 + \dot{x}_1\dot{x}_3 + \dot{x}_2\dot{x}_3 + \exp(2x_1) + \exp(x_2 - x_1) + \exp(x_3 - x_2). \quad (5)$$

Hamiltonian

The momenta, $p_I := \partial L/\partial \dot{x}_I$, are:

$$p_1 = \dot{x}_2 + \dot{x}_3, \quad p_2 = \dot{x}_1 + \dot{x}_3, \quad p_3 = \dot{x}_1 + \dot{x}_2. \tag{6}$$

The Hamiltonian of the system:

$$\begin{aligned} H \quad &:= p_I \dot{x}_I - L = \frac{1}{2}(p_1 p_2 + p_1 p_3 + p_2 p_3) \\ &- \frac{1}{4}(p_1^2 + p_2^2 + p_3^2) - \exp(2x_1) - \exp(x_2 - x_1) - \exp(x_3 - x_2), \end{aligned} \tag{7}$$

which due to (6) and (4) leads to the dynamical constraint:

$$H = 0. \tag{8}$$

Hamilton's equations

The Hamilton equations have the following explicit form:

$$
\begin{aligned}
\dot{x}_1 &= \frac{1}{2}(-p_1 + p_2 + p_3), && (9)\\
\dot{x}_2 &= \frac{1}{2}(p_1 - p_2 + p_3), && (10)\\
\dot{x}_3 &= \frac{1}{2}(p_1 + p_2 - p_3), && (11)\\
\dot{p}_1 &= 2\exp(2x_1) - \exp(x_2 - x_1), && (12)\\
\dot{p}_2 &= \exp(x_2 - x_1) - \exp(x_3 - x_2), && (13)\\
\dot{p}_3 &= \exp(x_3 - x_2), && (14)\\
H &= 0. && (15)
\end{aligned}
$$

One may show that Lagrangian and Hamiltonian formulations are completely equivalent. Analytical solution to this 6-dimensional nonlinear coupled system of equations are unknown.

Hamilton's equations

The Hamilton equations have the following explicit form:

$$\dot{x}_1 = \frac{1}{2}(-p_1 + p_2 + p_3), \tag{9}$$

$$\dot{x}_2 = \frac{1}{2}(p_1 - p_2 + p_3), \tag{10}$$

$$\dot{x}_3 = \frac{1}{2}(p_1 + p_2 - p_3), \tag{11}$$

$$\dot{p}_1 = 2\exp(2x_1) - \exp(x_2 - x_1), \tag{12}$$

$$\dot{p}_2 = \exp(x_2 - x_1) - \exp(x_3 - x_2), \tag{13}$$

$$\dot{p}_3 = \exp(x_3 - x_2), \tag{14}$$

$$H = 0. \tag{15}$$

One may show that Lagrangian and Hamiltonian formulations are completely equivalent. Analytical solution to this 6-dimensional nonlinear coupled system of equations are unknown.

Dynamical systems method

- The local geometry of the phase space is characterized by the nature and position of its critical points. These points are locations where the derivatives of all the dynamical variables vanish.
- The set of all critical points and their characteristic, given by the properties of the Jacobian matrix of the linearized equations at those points, may provide a qualitative description of a given dynamical system.
- The above situation is specific to the case when a fixed point is of the hyperbolic type. In the case of the nonhyperbolic fixed point, linearized vector field at the fixed point cannot be used to specify completely local properties of the phase space.

Dynamical systems method

- The local geometry of the phase space is characterized by the nature and position of its critical points. These points are locations where the derivatives of all the dynamical variables vanish.
- The set of all critical points and their characteristic, given by the properties of the Jacobian matrix of the linearized equations at those points, may provide a qualitative description of a given dynamical system.
- The above situation is specific to the case when a fixed point is of the hyperbolic type. In the case of the nonhyperbolic fixed point, linearized vector field at the fixed point cannot be used to specify completely local properties of the phase space.

Dynamical systems method

- The local geometry of the phase space is characterized by the nature and position of its critical points. These points are locations where the derivatives of all the dynamical variables vanish.
- The set of all critical points and their characteristic, given by the properties of the Jacobian matrix of the linearized equations at those points, may provide a qualitative description of a given dynamical system.
- The above situation is specific to the case when a fixed point is of the hyperbolic type. In the case of the nonhyperbolic fixed point, linearized vector field at the fixed point cannot be used to specify completely local properties of the phase space.

Dynamical systems analysis (cont)

The set of critical points fulfills the following conditions:

$$p_1 = 0 = p_2 = p_3, \tag{16}$$

$$x_1 \to -\infty, \; x_2 \to -\infty, \; x_3 \to -\infty, \tag{17}$$

$$x_3 < x_2 < x_1 < 0. \tag{18}$$

One may easily verify that this set satisfies the Hamiltonian constraint.
Thus the set of critical points S_B is given by

$$S_B : \; = \; \{(x_1, x_2, x_3, p_1, p_2, p_3) \in \bar{\mathbb{R}}^6 \mid (x_1 \to -\infty, \; x_2 \to -\infty, \; x_3 \to -\infty) \wedge (x_3 < x_2 < x_1 < 0); \;\; p_1 = 0 = p_2 = p_3\}, \tag{19}$$

where $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

Dynamical systems analysis (cont)

The set of critical points fulfills the following conditions:

$$p_1 = 0 = p_2 = p_3, \tag{16}$$

$$x_1 \to -\infty,\ x_2 \to -\infty,\ x_3 \to -\infty, \tag{17}$$

$$x_3 < x_2 < x_1 < 0. \tag{18}$$

One may easily verify that this set satisfies the Hamiltonian constraint. Thus the set of critical points S_B is given by

$$\begin{aligned} S_B : &= \{(x_1, x_2, x_3, p_1, p_2, p_3) \in \bar{\mathbb{R}}^6 \mid (x_1 \to -\infty,\ x_2 \to -\infty,\ x_3 \to -\infty) \\ &\wedge (x_3 < x_2 < x_1 < 0);\ \ p_1 = 0 = p_2 = p_3\}, \end{aligned} \tag{19}$$

where $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

Dynamical systems analysis (cont)

The Jacobian (at any point of the set S_B):

$$J = \begin{pmatrix} 0 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial associated with Jacobian J reads: $P(\lambda) = \lambda^6$, so the eigenvalues are the following: $(0, 0, 0, 0, 0, 0)$. Since the real parts of all eigenvalues of the Jacobian are equal to zero, the set S_B consists of nonhyperbolic fixed points.

Dynamical systems analysis (cont)

The Jacobian (at any point of the set S_B):

$$J = \begin{pmatrix} 0 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial associated with Jacobian J reads: $P(\lambda) = \lambda^6$, so the eigenvalues are the following: $(0, 0, 0, 0, 0, 0)$.
Since the real parts of all eigenvalues of the Jacobian are equal to zero, the set S_B consists of nonhyperbolic fixed points.

Dynamical systems analysis (cont)

The Jacobian (at any point of the set S_B):

$$J = \begin{pmatrix} 0 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial associated with Jacobian J reads: $P(\lambda) = \lambda^6$, so the eigenvalues are the following: $(0, 0, 0, 0, 0, 0)$. Since the real parts of all eigenvalues of the Jacobian are equal to zero, the set S_B consists of nonhyperbolic fixed points.

Dynamical systems analysis (cont)

Summary:

1. We are dealing with the nonhyperbolic type of critical points. Thus, getting insight into the structure of the space of orbits near such points requires an examination of the exact form of the vector field.
2. The phase space is higher dimensional.
3. The set of critical points S_B is not a set of isolated points, but a 3-dimensional continuous subspace of $\bar{\mathbb{R}}^6$.
4. The critical subspace S_B is situated in an asymptotic region of phase space with infinite values of its variables.

Dynamical systems analysis (cont)

Summary:

1. We are dealing with the nonhyperbolic type of critical points. Thus, getting insight into the structure of the space of orbits near such points requires an examination of the exact form of the vector field.
2. The phase space is higher dimensional.
3. The set of critical points S_B is not a set of isolated points, but a 3-dimensional continuous subspace of $\bar{\mathbb{R}}^6$.
4. The critical subspace S_B is situated in an asymptotic region of phase space with infinite values of its variables.

Dynamical systems analysis (cont)

Summary:

1. We are dealing with the nonhyperbolic type of critical points. Thus, getting insight into the structure of the space of orbits near such points requires an examination of the exact form of the vector field.
2. The phase space is higher dimensional.
3. The set of critical points S_B is not a set of isolated points, but a 3-dimensional continuous subspace of $\bar{\mathbb{R}}^6$.
4. The critical subspace S_B is situated in an asymptotic region of phase space with infinite values of its variables.

Dynamical systems analysis (cont)

Summary:

1. We are dealing with the nonhyperbolic type of critical points. Thus, getting insight into the structure of the space of orbits near such points requires an examination of the exact form of the vector field.
2. The phase space is higher dimensional.
3. The set of critical points S_B is not a set of isolated points, but a 3-dimensional continuous subspace of $\bar{\mathbb{R}}^6$.
4. The critical subspace S_B is situated in an asymptotic region of phase space with infinite values of its variables.

Nonhyperbolicity[8]

Are the nonhyperbolic critical points directly connected with the chaotic dynamics?

[8] E. Czuchry and WP, "Bianchi IX model: Reducing phase space," Phys. Rev. D **87** (2013) 084021;
E.Czuchry, J. Hell, and WP, 'Bianchi IX model: Comparing diagonal and nondiagonal cases', in preparation.

Semi-classical Bianchi IX model

In the Misner like parametrization the Hamiltonian for the vacuum Bianchi type models reads[9]

$$\mathcal{H} = \mathcal{N}(t)\left(\frac{2\pi G}{3c^2a^3}\left(a^2p_a^2 - p_+^2 - p_-^2\right) - \frac{c^4}{32\pi G}aW_n(\beta_\pm)\right) \approx 0, \quad (20)$$

where $(a, \beta\pm; p_a, p_\pm)$ are canonical variables.

Well known homogeneous models can be obtained as follows:

- FRW, by taking $W_n(\beta_\pm) = 0$ and $p_\pm = 0$;
- Bianchi-I, corresponds to $W_n(\beta_\pm) = 0$;
- Bianchi-II, has $W_n(\beta_\pm) = n^2 e^{8\beta_+}$ and $n > 0$.

[9] H. Bergeron, E. Czuchry, J-P. Gazeau, P. Małkiewicz, and WP, *'Quantum Bianchi IX model'*, in preparation.

Semi-classical Bianchi IX model

In the Misner like parametrization the Hamiltonian for the vacuum Bianchi type models reads[9]

$$\mathcal{H} = \mathcal{N}(t)\left(\frac{2\pi G}{3c^2a^3}\left(a^2p_a^2 - p_+^2 - p_-^2\right) - \frac{c^4}{32\pi G}aW_n(\beta_\pm)\right) \approx 0, \quad (20)$$

where $(a, \beta\pm; p_a, p_\pm)$ are canonical variables.
Well known homogeneous models can be obtained as follows:

- FRW, by taking $W_n(\beta_\pm) = 0$ and $p_\pm = 0$;
- Bianchi-I, corresponds to $W_n(\beta_\pm) = 0$;
- Bianchi-II, has $W_n(\beta_\pm) = n^2e^{8\beta_+}$ and $n > 0$.

[9] H. Bergeron, E. Czuchry, J-P. Gazeau, P. Małkiewicz, and WP, *'Quantum Bianchi IX model'*, in preparation.

Classical Hamiltonian

The Bianchi IX model is defined by

$$W_n(\beta_\pm) = n^2 e^{-4\beta_+}\left(\left(e^{6\beta_+} - 2\cosh(2\sqrt{3}\beta_-)\right)^2 - 4\right), \; n > 0. \quad (21)$$

The potential W_n is bounded from below and reaches its (absolute) minimal value at $\beta_\pm = 0$, with $W_n(0) = -3n^2$.
W_n has $\mathbb{C}_{3v}$ symmetry and is asymptotically confined except for three directions:

(i) $\beta_- = 0$, $\beta_+ \to -\infty$,

(ii) $\beta_+ = \beta_-/\sqrt{3}$, $\beta_- \to +\infty$,

(iii) $\beta_+ = -\beta_-/\sqrt{3}$, $\beta_- \to -\infty$,

where $W_n \to 0$.

Classical Hamiltonian

The Bianchi IX model is defined by

$$W_n(\beta_\pm) = n^2 e^{-4\beta_+} \left(\left(e^{6\beta_+} - 2\cosh(2\sqrt{3}\beta_-) \right)^2 - 4 \right), \;\; n > 0. \quad (21)$$

The potential W_n is bounded from below and reaches its (absolute) minimal value at $\beta_\pm = 0$, with $W_n(0) = -3n^2$.

W_n has $\mathbb{C}_{3v}$ symmetry and is asymptotically confined except for three directions:

(i) $\beta_- = 0$, $\beta_+ \to -\infty$,

(ii) $\beta_+ = \beta_-/\sqrt{3}$, $\beta_- \to +\infty$,

(iii) $\beta_+ = -\beta_-/\sqrt{3}$, $\beta_- \to -\infty$,

where $W_n \to 0$.

Classical Hamiltonian

The Bianchi IX model is defined by

$$W_n(\beta_\pm) = n^2 e^{-4\beta_+}\left(\left(e^{6\beta_+} - 2\cosh(2\sqrt{3}\beta_-)\right)^2 - 4\right), \; n > 0. \quad (21)$$

The potential W_n is bounded from below and reaches its (absolute) minimal value at $\beta_\pm = 0$, with $W_n(0) = -3n^2$.
W_n has $\mathbb{C}_{3v}$ symmetry and is asymptotically confined except for three directions:

(i) $\beta_- = 0, \beta_+ \to -\infty$,

(ii) $\beta_+ = \beta_-/\sqrt{3}, \beta_- \to +\infty$,

(iii) $\beta_+ = -\beta_-/\sqrt{3}, \beta_- \to -\infty$,

where $W_n \to 0$.

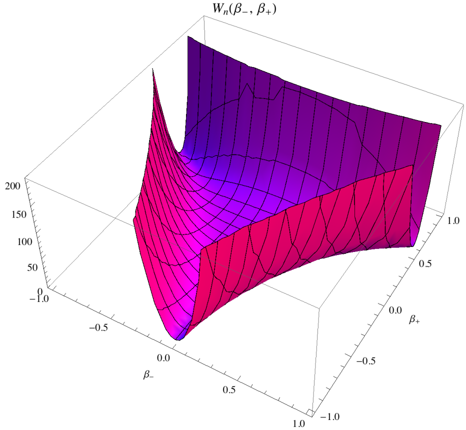

Figure: The plot of W_n near its minimum. Boundedness from below, confinement aspects, and C_{3v} symmetry are illustrated.

Classical Hamiltonian (cont)

Redefining the phase space variables, to suggest possible approximation, by introducing the canonical pair $(q = a^{3/2}, p = 2p_a/(3\sqrt{a}))$:

$$\mathcal{H} = \mathcal{N}(t) \left(\frac{2\pi G}{3c^2} \left(\frac{9}{4} p^2 - \frac{p_+^2 + p_-^2}{q^2} \right) - \frac{c^4}{32\pi G} q^{2/3} W_n(\beta_\pm) \right) . \qquad (22)$$

It results from Eq. (22) that near the singularity, $q = 0$, we may treat q as heavy degree of freedom, and $\beta_\pm$ as light degrees of freedom. It is so because 'mass' of the $\beta_\pm$ behaves as q^2, while 'mass' of q is fixed. Therefore, we may quantize our system by using an adiabatic approximation.

Classical Hamiltonian (cont)

Redefining the phase space variables, to suggest possible approximation, by introducing the canonical pair $(q = a^{3/2}, p = 2p_a/(3\sqrt{a}))$:

$$\mathcal{H} = \mathcal{N}(t)\left(\frac{2\pi G}{3c^2}\left(\frac{9}{4}p^2 - \frac{p_+^2 + p_-^2}{q^2}\right) - \frac{c^4}{32\pi G}q^{2/3}W_n(\beta_\pm)\right). \quad (22)$$

It results from Eq. (22) that near the singularity, $q = 0$, we may treat q as heavy degree of freedom, and $\beta_\pm$ as light degrees of freedom. It is so because 'mass' of the $\beta_\pm$ behaves as q^2, while 'mass' of q is fixed. Therefore, we may quantize our system by using an adiabatic approximation.

Classical Hamiltonian (cont)

Redefining the phase space variables, to suggest possible approximation, by introducing the canonical pair $(q = a^{3/2}, p = 2p_a/(3\sqrt{a}))$:

$$\mathcal{H} = \mathcal{N}(t)\left(\frac{2\pi G}{3c^2}\left(\frac{9}{4}p^2 - \frac{p_+^2 + p_-^2}{q^2}\right) - \frac{c^4}{32\pi G}q^{2/3}W_n(\beta_\pm)\right). \quad (22)$$

It results from Eq. (22) that near the singularity, $q = 0$, we may treat q as heavy degree of freedom, and $\beta_\pm$ as light degrees of freedom. It is so because 'mass' of the $\beta_\pm$ behaves as q^2, while 'mass' of q is fixed. Therefore, we may quantize our system by using an adiabatic approximation.

Classical Hamiltonian (cont)

For the purpose of the adiabatic quantization:

$$\mathcal{H} = \mathcal{N}(t)\left(\frac{3\pi G}{2c^2}p^2 - \mathcal{H}_\pm\right), \tag{23}$$

where

$$\mathcal{H}_\pm := \frac{2\pi G}{3c^2 q^2}(p_+^2 + p_-^2) + \frac{c^4}{32\pi G}q^{2/3}W_n(\beta_\pm)\,. \tag{24}$$

- $\beta_\pm = 0 = p_\pm$ corresponds to the classical ground state of the anisotropy Hamiltonian $\mathcal{H}_\pm$. Thus, FRW may be treated as a special case of Bianchi-IX , where the anisotropy degrees of freedom are frozen in their (classical) ground state.
- We cannot quantize the FRW model alone, because we should take into account the effect of quantum 'zero point energy' generated by the quantized anisotropy degrees of freedom of the Bianchi-IX model.

Classical Hamiltonian (cont)

For the purpose of the adiabatic quantization:

$$\mathcal{H} = \mathcal{N}(t)\left(\frac{3\pi G}{2c^2}p^2 - \mathcal{H}_\pm\right), \tag{23}$$

where

$$\mathcal{H}_\pm := \frac{2\pi G}{3c^2q^2}(p_+^2 + p_-^2) + \frac{c^4}{32\pi G}q^{2/3}W_n(\beta_\pm)\,. \tag{24}$$

- $\beta_\pm = 0 = p_\pm$ corresponds to the classical ground state of the anisotropy Hamiltonian $\mathcal{H}_\pm$. Thus, FRW may be treated as a special case of Bianchi-IX , where the anisotropy degrees of freedom are frozen in their (classical) ground state.
- We cannot quantize the FRW model alone, because we should take into account the effect of quantum 'zero point energy' generated by the quantized anisotropy degrees of freedom of the Bianchi-IX model.

Classical Hamiltonian (cont)

For the purpose of the adiabatic quantization:

$$\mathcal{H} = \mathcal{N}(t) \left(\frac{3\pi G}{2c^2} p^2 - \mathcal{H}_\pm \right), \tag{23}$$

where

$$\mathcal{H}_\pm := \frac{2\pi G}{3c^2 q^2} (p_+^2 + p_-^2) + \frac{c^4}{32\pi G} q^{2/3} W_n(\beta_\pm) \,. \tag{24}$$

- $\beta_\pm = 0 = p_\pm$ corresponds to the classical ground state of the anisotropy Hamiltonian $\mathcal{H}_\pm$. Thus, FRW may be treated as a special case of Bianchi-IX , where the anisotropy degrees of freedom are frozen in their (classical) ground state.
- We cannot quantize the FRW model alone, because we should take into account the effect of quantum 'zero point energy' generated by the quantized anisotropy degrees of freedom of the Bianchi-IX model.

Quantum Hamiltonian

In what follows we apply the modified Dirac quantization method:

- quantizing $\mathcal{H}$ (all degrees of freedom) to get $\hat{\mathcal{H}}$,
- finding semi-classical expression $\check{\mathcal{H}}$ of $\hat{\mathcal{H}}$,
- making adiabatic approximation,
- implementing constraint $\check{\mathcal{H}} = 0$.

Since $(q, p) \in \mathbb{R}_+ \times \mathbb{R}$ and $(\beta_\pm, p_\pm) \in \mathbb{R}^4$:

- we apply affine coherent states quantization to (q, p),
- we apply canonical quantization to $(\beta_\pm, p_\pm)$

Quantum Hamiltonian

In what follows we apply the modified Dirac quantization method:

- quantizing $\mathcal{H}$ (all degrees of freedom) to get $\hat{\mathcal{H}}$,
- finding semi-classical expression $\check{\mathcal{H}}$ of $\hat{\mathcal{H}}$,
- making adiabatic approximation,
- implementing constraint $\check{\mathcal{H}} = 0$.

Since $(q, p) \in \mathbb{R}_+ \times \mathbb{R}$ and $(\beta_\pm, p_\pm) \in \mathbb{R}^4$:

- we apply affine coherent states quantization to (q, p),
- we apply canonical quantization to $(\beta_\pm, p_\pm)$

Quantum Hamiltonian (cont)

The quantum Hamiltonian $\hat{\mathcal{H}}$ reads (we put $\mathcal{N} = 1$):

$$\hat{\mathcal{H}} = \frac{3\pi G}{2c^2}\left(\hat{p}^2 + \frac{\hbar^2 \mathfrak{K}_1}{\hat{q}^2}\right) - \frac{2\pi G}{3c^2}\mathfrak{K}_2 \frac{\hat{p}_+^2 + \hat{p}_-^2}{\hat{q}^2} - \frac{c^4}{32\pi G}\mathfrak{K}_3\, \hat{q}^{2/3} W_n(\hat{\beta}_\pm)\,, \tag{25}$$

where the $\mathfrak{K}_i$:

$$\mathfrak{K}_1 = \frac{1}{4}\left(1 + \frac{K_0(\nu)}{K_1(\nu)}\right), \quad \mathfrak{K}_2 = \left(\frac{K_2(\nu)}{K_1(\nu)}\right)^2 \quad \mathfrak{K}_3 = \frac{K_{5/3}(\nu)}{K_1(\nu)^{1/3} K_2(\nu)^{2/3}}\,, \tag{26}$$

and where the $K_\alpha(\nu)$ are modified Bessel functions.
$\hat{p}_\pm = -i\hbar\partial_{\beta_\pm}$, and $\hat{\beta}_\pm$ defined as $\beta_\pm$, acting on $L^2(\mathbb{R}^2, d\beta_+ d\beta_-)$;
$\hat{p} = -i\hbar\partial_q$, and $\hat{q}$ defined as q, acting on $L^2(\mathbb{R}_+, dq)$.

Semi-classical approximation

We have

$$\hat{\mathcal{H}}_\pm(q) = \frac{2\pi G}{3c^2}\mathfrak{K}_2 \frac{\hat{p}_+^2 + \hat{p}_-^2}{q^2} + \frac{c^4}{32\pi G}\mathfrak{K}_3\, q^{2/3} W_n(\beta_\pm)\,. \tag{27}$$

Due to the harmonic behavior of W_n near its minimum:

$$W_n(\beta_\pm) \simeq -3n^2 + 24n^2(\beta_+^2 + \beta_-^2) + o(\beta_\pm^2)\,, \tag{28}$$

we approximate the eigen-energies $E_\pm^{(N)}$ of $\hat{\mathcal{H}}_\pm$ as follows:

$$E_\pm^{(N)}(q) \simeq -\frac{3c^4}{32\pi G}\mathfrak{K}_3 q^{2/3} n^2 + \frac{\hbar c}{q^{2/3}} n\sqrt{2\mathfrak{K}_2\mathfrak{K}_3}\,(N+1)\,, \tag{29}$$

where $N = 0, 1, \ldots$.

Semi-classical approximation

We have

$$\hat{\mathcal{H}}_{\pm}(q) = \frac{2\pi G}{3c^2}\mathfrak{K}_2 \frac{\hat{p}_+^2 + \hat{p}_-^2}{q^2} + \frac{c^4}{32\pi G}\,\mathfrak{K}_3\, q^{2/3} W_n(\beta_{\pm})\,. \qquad (27)$$

Due to the harmonic behavior of W_n near its minimum:

$$W_n(\beta_{\pm}) \simeq -3n^2 + 24n^2(\beta_+^2 + \beta_-^2) + o(\beta_{\pm}^2)\,, \qquad (28)$$

we approximate the eigen-energies $E_{\pm}^{(N)}$ of $\hat{\mathcal{H}}_{\pm}$ as follows:

$$E_{\pm}^{(N)}(q) \simeq -\frac{3c^4}{32\pi G}\mathfrak{K}_3 q^{2/3} n^2 + \frac{\hbar c}{q^{2/3}} n\sqrt{2\mathfrak{K}_2\mathfrak{K}_3}\,(N+1)\,, \qquad (29)$$

where $N = 0, 1, \ldots$.

Semi-classical approximation

We have

$$\hat{\mathcal{H}}_\pm(q) = \frac{2\pi G}{3c^2}\mathfrak{K}_2\frac{\hat{p}_+^2 + \hat{p}_-^2}{q^2} + \frac{c^4}{32\pi G}\mathfrak{K}_3\, q^{2/3} W_n(\beta_\pm)\,. \qquad (27)$$

Due to the harmonic behavior of W_n near its minimum:

$$W_n(\beta_\pm) \simeq -3n^2 + 24n^2(\beta_+^2 + \beta_-^2) + o(\beta_\pm^2)\,, \qquad (28)$$

we approximate the eigen-energies $E_\pm^{(N)}$ of $\hat{\mathcal{H}}_\pm$ as follows:

$$E_\pm^{(N)}(q) \simeq -\frac{3c^4}{32\pi G}\mathfrak{K}_3 q^{2/3} n^2 + \frac{\hbar c}{q^{2/3}} n\sqrt{2\mathfrak{K}_2\mathfrak{K}_3}\,(N+1)\,, \qquad (29)$$

where $N = 0, 1, \ldots$.

Semi-classical approximation (cont)

The Hamiltonian $\hat{\mathcal{H}}$ is now replaced by the one with frozen anisotropy degrees of freedom in some eigen state evolving adiabatically:

$$\hat{\mathcal{H}}_{av} = \mathcal{N}(t)\left(\frac{3\pi G}{2c^2}\left(\hat{p}^2 + \frac{\hbar^2 \mathfrak{K}_1}{\hat{q}^2}\right) - E_{\pm}^{(N)}(\hat{q})\right) . \tag{30}$$

The semi-classical expressions with affine CS, is defined as

$$\check{\mathcal{H}}_{av}(q,p) = \langle \lambda q, p | \hat{\mathcal{H}}_{av} | \lambda q, p \rangle , \tag{31}$$

where $\lambda := K_0(\nu)/K_2(\nu)$ is chosen to get $\langle \lambda q, p | \hat{q} | \lambda q, p \rangle = q$ and $\langle \lambda q, p | \hat{p} | \lambda q, p \rangle = p$.
Finally, we obtain

$$\check{\mathcal{H}}_{av} = \mathcal{N}(t)\left(\frac{3\pi G}{2c^2}\left(p^2 + \frac{\hbar^2 \mathfrak{K}_4}{q^2}\right) + \frac{3c^4}{32\pi G}\mathfrak{K}_5 q^{2/3} n^2 - \frac{\hbar c}{q^{2/3}}\mathfrak{K}_6 n(N+1)\right) , \tag{32}$$

where $\mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_6$ are numerical factors.

Semi-classical approximation (cont)

The Hamiltonian $\hat{\mathcal{H}}$ is now replaced by the one with frozen anisotropy degrees of freedom in some eigen state evolving adiabatically:

$$\hat{\mathcal{H}}_{av} = \mathcal{N}(t)\left(\frac{3\pi G}{2c^2}\left(\hat{p}^2 + \frac{\hbar^2 \mathfrak{K}_1}{\hat{q}^2}\right) - E_{\pm}^{(N)}(\hat{q})\right) . \qquad (30)$$

The semi-classical expressions with affine CS, is defined as

$$\check{\mathcal{H}}_{av}(q,p) = \langle \lambda q, p | \hat{\mathcal{H}}_{av} | \lambda q, p \rangle , \qquad (31)$$

where $\lambda := K_0(\nu)/K_2(\nu)$ is chosen to get $\langle \lambda q, p|\hat{q}|\lambda q, p\rangle = q$ and $\langle \lambda q, p|\hat{p}|\lambda q, p\rangle = p$.
Finally, we obtain

$$\check{\mathcal{H}}_{av} = \mathcal{N}(t)\left(\frac{3\pi G}{2c^2}\left(p^2 + \frac{\hbar^2 \mathfrak{K}_4}{q^2}\right) + \frac{3c^4}{32\pi G}\mathfrak{K}_5 q^{2/3} n^2 - \frac{\hbar c}{q^{2/3}}\mathfrak{K}_6 n(N+1)\right) , \qquad (32)$$

where $\mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_6$ are numerical factors.

Semi-classical approximation (cont)

The Hamiltonian $\hat{\mathcal{H}}$ is now replaced by the one with frozen anisotropy degrees of freedom in some eigen state evolving adiabatically:

$$\hat{\mathcal{H}}_{av} = \mathcal{N}(t)\left(\frac{3\pi G}{2c^2}\left(\hat{p}^2 + \frac{\hbar^2 \mathfrak{K}_1}{\hat{q}^2}\right) - E_{\pm}^{(N)}(\hat{q})\right) . \tag{30}$$

The semi-classical expressions with affine CS, is defined as

$$\check{\mathcal{H}}_{av}(q,p) = \langle \lambda q, p | \hat{\mathcal{H}}_{av} | \lambda q, p \rangle \, , \tag{31}$$

where $\lambda := K_0(\nu)/K_2(\nu)$ is chosen to get $\langle \lambda q, p|\hat{q}|\lambda q, p\rangle = q$ and $\langle \lambda q, p|\hat{p}|\lambda q, p\rangle = p$.
Finally, we obtain

$$\check{\mathcal{H}}_{av} = \mathcal{N}(t)\left(\frac{3\pi G}{2c^2}\left(p^2 + \frac{\hbar^2 \mathfrak{K}_4}{q^2}\right) + \frac{3c^4}{32\pi G}\mathfrak{K}_5 q^{2/3} n^2 - \frac{\hbar c}{q^{2/3}}\mathfrak{K}_6 n(N+1)\right) , \tag{32}$$

where $\mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_6$ are numerical factors.

Imposition of Hamiltonian constraint:

The constraint $\check{\mathcal{H}}_{av}(q,p) = 0$ reads

$$\frac{\dot{a}^2}{a^2} + k\frac{c^2}{a^2} + \mathfrak{s}_P^2 c^2 \frac{\mathfrak{K}_4}{a^6} = \frac{8\pi G}{3c^2}\rho(a)\,, \tag{33}$$

where

$$\mathfrak{s}_P := 2\pi G\hbar\, c^{-3}, \quad k := \frac{\mathfrak{K}_5 n^2}{4}, \quad \rho(a) := \hbar c(N+1)\frac{n\mathfrak{K}_6}{a^4}\,. \tag{34}$$

The main features of this quantum model:

- anisotropy degrees of freedom produce radiation-like energy density $\rho(a)$ (we consider vacuum BIX model);
- new repulsive potential term with a^{-6} generated by the affine CS quantization;
- curvature-like term kc^2a^{-2}, present in the classical BIX model.

Imposition of Hamiltonian constraint:

The constraint $\check{\mathcal{H}}_{av}(q,p) = 0$ reads

$$\frac{\dot{a}^2}{a^2} + k\frac{c^2}{a^2} + \mathfrak{s}_P^2 c^2 \frac{\mathfrak{K}_4}{a^6} = \frac{8\pi G}{3c^2}\rho(a)\,, \tag{33}$$

where

$$\mathfrak{s}_P := 2\pi G\hbar\, c^{-3}, \quad k := \frac{\mathfrak{K}_5 n^2}{4}, \quad \rho(a) := \hbar c(N+1)\frac{n\mathfrak{K}_6}{a^4}\,. \tag{34}$$

The main features of this quantum model:

- anisotropy degrees of freedom produce radiation-like energy density $\rho(a)$ (we consider vacuum BIX model);
- new repulsive potential term with a^{-6} generated by the affine CS quantization;
- curvature-like term kc^2a^{-2}, present in the classical BIX model.

Imposition of Hamiltonian constraint:

The constraint $\check{\mathcal{H}}_{av}(q,p) = 0$ reads

$$\frac{\dot{a}^2}{a^2} + k\frac{c^2}{a^2} + \mathfrak{s}_P^2 c^2 \frac{\mathfrak{K}_4}{a^6} = \frac{8\pi G}{3c^2}\rho(a)\,, \qquad (33)$$

where

$$\mathfrak{s}_P := 2\pi G\hbar\, c^{-3}, \quad k := \frac{\mathfrak{K}_5 n^2}{4}, \quad \rho(a) := \hbar c(N+1)\frac{n\mathfrak{K}_6}{a^4}\,. \qquad (34)$$

The main features of this quantum model:

- anisotropy degrees of freedom produce radiation-like energy density $\rho(a)$ (we consider vacuum BIX model);
- new repulsive potential term with a^{-6} generated by the affine CS quantization;
- curvature-like term $kc^2 a^{-2}$, present in the classical BIX model.

Imposition of Hamiltonian constraint:

The constraint $\check{\mathcal{H}}_{av}(q,p) = 0$ reads

$$\frac{\dot{a}^2}{a^2} + k\frac{c^2}{a^2} + \mathfrak{s}_P^2 c^2 \frac{\mathfrak{K}_4}{a^6} = \frac{8\pi G}{3c^2}\rho(a)\,, \qquad (33)$$

where

$$\mathfrak{s}_P := 2\pi G\hbar\, c^{-3}, \quad k := \frac{\mathfrak{K}_5 n^2}{4}, \quad \rho(a) := \hbar c(N+1)\frac{n\mathfrak{K}_6}{a^4}\,. \qquad (34)$$

The main features of this quantum model:

- anisotropy degrees of freedom produce radiation-like energy density $\rho(a)$ (we consider vacuum BIX model);
- new repulsive potential term with a^{-6} generated by the affine CS quantization;
- curvature-like term kc^2a^{-2}, present in the classical BIX model.

Imposition of Hamiltonian constraint:

The constraint $\check{\mathcal{H}}_{av}(q,p) = 0$ reads

$$\frac{\dot{a}^2}{a^2} + k\frac{c^2}{a^2} + \mathfrak{s}_P^2 c^2 \frac{\mathfrak{K}_4}{a^6} = \frac{8\pi G}{3c^2}\rho(a)\,, \qquad (33)$$

where

$$\mathfrak{s}_P := 2\pi G\hbar\, c^{-3}, \quad k := \frac{\mathfrak{K}_5 n^2}{4}, \quad \rho(a) := \hbar c(N+1)\frac{n\mathfrak{K}_6}{a^4}\,. \qquad (34)$$

The main features of this quantum model:

- anisotropy degrees of freedom produce radiation-like energy density $\rho(a)$ (we consider vacuum BIX model);
- new repulsive potential term with a^{-6} generated by the affine CS quantization;
- curvature-like term $kc^2 a^{-2}$, present in the classical BIX model.

Resolution of singularity

Equation (33) can be rewritten as

$$kc^2 + \mathfrak{s}_P^2 c^2 \frac{\mathfrak{K}_4}{a^4} - \frac{8\pi G}{3c^2} a^2 \rho(a) \leq 0, \tag{35}$$

which defines allowed values of scale factor $a \in [a_-, a_+]$. Thus, the semi-classical trajectories are bounded:

from below

$$\frac{\mathfrak{s}_P}{a_-^2} = \frac{2\mathfrak{K}_6}{3\mathfrak{K}_4} n(N+1) \left(1 + \sqrt{1 - \frac{f(\nu)}{(N+1)^2}}\right) \tag{36}$$

and from above

$$\frac{\mathfrak{s}_P}{a_+^2} = \frac{2\mathfrak{K}_6}{3\mathfrak{K}_4} n(N+1) \left(1 - \sqrt{1 - \frac{f(\nu)}{(N+1)^2}}\right). \tag{37}$$

Resolution of singularity

Equation (33) can be rewritten as

$$kc^2 + \mathfrak{s}_P^2 c^2 \frac{\mathfrak{K}_4}{a^4} - \frac{8\pi G}{3c^2} a^2 \rho(a) \leq 0, \tag{35}$$

which defines allowed values of scale factor $a \in [a_-, a_+]$. Thus, the semi-classical trajectories are bounded:
from below

$$\frac{\mathfrak{s}_P}{a_-^2} = \frac{2\mathfrak{K}_6}{3\mathfrak{K}_4} n(N+1) \left(1 + \sqrt{1 - \frac{f(\nu)}{(N+1)^2}}\right) \tag{36}$$

and from above

$$\frac{\mathfrak{s}_P}{a_+^2} = \frac{2\mathfrak{K}_6}{3\mathfrak{K}_4} n(N+1) \left(1 - \sqrt{1 - \frac{f(\nu)}{(N+1)^2}}\right). \tag{37}$$

Resolution of singularity

Equation (33) can be rewritten as

$$kc^2 + \mathfrak{s}_P^2 c^2 \frac{\mathfrak{K}_4}{a^4} - \frac{8\pi G}{3c^2} a^2 \rho(a) \leq 0, \tag{35}$$

which defines allowed values of scale factor $a \in [a_-, a_+]$. Thus, the semi-classical trajectories are bounded:
from below

$$\frac{\mathfrak{s}_P}{a_-^2} = \frac{2\mathfrak{K}_6}{3\mathfrak{K}_4} n(N+1) \left(1 + \sqrt{1 - \frac{f(\nu)}{(N+1)^2}}\right) \tag{36}$$

and from above

$$\frac{\mathfrak{s}_P}{a_+^2} = \frac{2\mathfrak{K}_6}{3\mathfrak{K}_4} n(N+1) \left(1 - \sqrt{1 - \frac{f(\nu)}{(N+1)^2}}\right). \tag{37}$$

Periodicity of trajectories

The semi-classical trajectories are periodic.
The oscillatory period T of the universe is

$$T = \frac{2\mathfrak{t}_P}{\sqrt{\mathfrak{K}_4}} (x_- x_+)^{-3/4} \left(\frac{x_+}{x_-} \right)^{-1/4} E \left(1 - \frac{x_+}{x_-} \right) , \tag{38}$$

where $\mathfrak{t}_P = \sqrt{\mathfrak{s}_P}/c$ is the Planck time, $x_\pm = \mathfrak{s}_P/a_\mp^2$, and E is the complete elliptic integral of the second kind.

Conclusions

Applying

- mixed procedure of quantization (CS and canonical),
- adiabatic approximation to the quantum Hamiltonian ,
- constraint $\mathcal{H} = 0$ at the semi-classical level,

it is possible to develop a quantum version of the classical Bianchi-IX model that looks like a modified FRW model.

The main features of this quantum model are:

- the transformation of the quantum energy due to anisotropy degrees of freedom into radiation-like term $\propto a^{-4}$
- new repulsive potential term $\propto a^{-6}$ generated by quantization, responsible for the resolution of the singularity
- a curvature-like term $\propto a^{-2}$ (already present in the classical Bianchi-IX treatment)

Conclusions

Applying

- mixed procedure of quantization (CS and canonical),
- adiabatic approximation to the quantum Hamiltonian ,
- constraint $\mathcal{H} = 0$ at the semi-classical level,

it is possible to develop a quantum version of the classical Bianchi-IX model that looks like a modified FRW model.

The main features of this quantum model are:

- the transformation of the quantum energy due to anisotropy degrees of freedom into radiation-like term $\propto a^{-4}$
- new repulsive potential term $\propto a^{-6}$ generated by quantization, responsible for the resolution of the singularity
- a curvature-like term $\propto a^{-2}$ (already present in the classical Bianchi-IX treatment)

Conclusions

Applying

- mixed procedure of quantization (CS and canonical),
- adiabatic approximation to the quantum Hamiltonian ,
- constraint $\mathcal{H} = 0$ at the semi-classical level,

it is possible to develop a quantum version of the classical Bianchi-IX model that looks like a modified FRW model.
The main features of this quantum model are:

- the transformation of the quantum energy due to anisotropy degrees of freedom into radiation-like term $\propto a^{-4}$
- new repulsive potential term $\propto a^{-6}$ generated by quantization, responsible for the resolution of the singularity
- a curvature-like term $\propto a^{-2}$ (already present in the classical Bianchi-IX treatment)

Conclusions

Applying

- mixed procedure of quantization (CS and canonical),
- adiabatic approximation to the quantum Hamiltonian ,
- constraint $\mathcal{H} = 0$ at the semi-classical level,

it is possible to develop a quantum version of the classical Bianchi-IX model that looks like a modified FRW model.

The main features of this quantum model are:

- the transformation of the quantum energy due to anisotropy degrees of freedom into radiation-like term $\propto a^{-4}$
- new repulsive potential term $\propto a^{-6}$ generated by quantization, responsible for the resolution of the singularity
- a curvature-like term $\propto a^{-2}$ (already present in the classical Bianchi-IX treatment)

Conclusions

Applying

- mixed procedure of quantization (CS and canonical),
- adiabatic approximation to the quantum Hamiltonian ,
- constraint $\mathcal{H} = 0$ at the semi-classical level,

it is possible to develop a quantum version of the classical Bianchi-IX model that looks like a modified FRW model.
The main features of this quantum model are:

- the transformation of the quantum energy due to anisotropy degrees of freedom into radiation-like term $\propto a^{-4}$
- new repulsive potential term $\propto a^{-6}$ generated by quantization, responsible for the resolution of the singularity
- a curvature-like term $\propto a^{-2}$ (already present in the classical Bianchi-IX treatment)

Conclusions

Applying

- mixed procedure of quantization (CS and canonical),
- adiabatic approximation to the quantum Hamiltonian ,
- constraint $\mathcal{H} = 0$ at the semi-classical level,

it is possible to develop a quantum version of the classical Bianchi-IX model that looks like a modified FRW model.
The main features of this quantum model are:

- the transformation of the quantum energy due to anisotropy degrees of freedom into radiation-like term $\propto a^{-4}$
- new repulsive potential term $\propto a^{-6}$ generated by quantization, responsible for the resolution of the singularity
- a curvature-like term $\propto a^{-2}$ (already present in the classical Bianchi-IX treatment)

Chaotic dynamics:

- Classical dynamics of the vacuum Bianchi IX model is chaotic[10]
- What about quantum dynamics[11]?
- Universality conjecture for energy levels distribution:
  - chaotic classical systems are characterized by the Gaussian like distribution describing the 'level repulsion' in quantum theory[12]:

$$P_{GOE}(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right). \quad (39)$$

  - regular classical systems are characterized by the Poisson like distribution describing 'level clustering' in quantum theory:

$$P_{Poisson}(s) = e^{-s}, \quad (40)$$

  where s is level-spacing.

[10] see, e.g., N. J. Cornish and J. J. Levin, "The Mixmaster universe is chaotic", Phys. Rev. Lett. **78** (1997) 998; "The Mixmaster universe: A Chaotic Farey tale", Phys. Rev. D **55** (1997) 7489

[11] J. Mielczarek and WP, work in progress.

[12] It may be described by the distribution based on Gaussian Orthogonal Ensemble of the random matrices.

Chaotic dynamics:

- Classical dynamics of the vacuum Bianchi IX model is chaotic[10]
- What about quantum dynamics[11]?
- Universality conjecture for energy levels distribution:
  - chaotic classical systems are characterized by the Gaussian like distribution describing the 'level repulsion' in quantum theory[12]:

$$P_{GOE}(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right). \tag{39}$$

  - regular classical systems are characterized by the Poisson like distribution describing 'level clustering' in quantum theory:

$$P_{Poisson}(s) = e^{-s}, \tag{40}$$

  where s is level-spacing.

[10] see, e.g., N. J. Cornish and J. J. Levin, "The Mixmaster universe is chaotic", Phys. Rev. Lett. **78** (1997) 998; "The Mixmaster universe: A Chaotic Farey tale", Phys. Rev. D **55** (1997) 7489

[11] J. Mielczarek and WP, work in progress.

[12] It may be described by the distribution based on Gaussian Orthogonal Ensemble of the random matrices.

Chaotic dynamics:

- Classical dynamics of the vacuum Bianchi IX model is chaotic[10]
- What about quantum dynamics[11]?
- Universality conjecture for energy levels distribution:
  - chaotic classical systems are characterized by the Gaussian like distribution describing the 'level repulsion' in quantum theory[12]:

$$P_{GOE}(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right). \tag{39}$$

  - regular classical systems are characterized by the Poisson like distribution describing 'level clustering' in quantum theory:

$$P_{Poisson}(s) = e^{-s}, \tag{40}$$

  where s is level-spacing.

[10] see, e.g., N. J. Cornish and J. J. Levin, "The Mixmaster universe is chaotic", Phys. Rev. Lett. **78** (1997) 998; "The Mixmaster universe: A Chaotic Farey tale", Phys. Rev. D **55** (1997) 7489

[11] J. Mielczarek and WP, work in progress.

[12] It may be described by the distribution based on Gaussian Orthogonal Ensemble of the random matrices.

Chaotic dynamics:

- Classical dynamics of the vacuum Bianchi IX model is chaotic[10]
- What about quantum dynamics[11]?
- Universality conjecture for energy levels distribution:
  - chaotic classical systems are characterized by the Gaussian like distribution describing the 'level repulsion' in quantum theory[12]:

$$P_{GOE}(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right). \tag{39}$$

  - regular classical systems are characterized by the Poisson like distribution describing 'level clustering' in quantum theory:

$$P_{Poisson}(s) = e^{-s}, \tag{40}$$

  where s is level-spacing.

[10] see, e.g., N. J. Cornish and J. J. Levin, "The Mixmaster universe is chaotic", Phys. Rev. Lett. **78** (1997) 998; "The Mixmaster universe: A Chaotic Farey tale", Phys. Rev. D **55** (1997) 7489

[11] J. Mielczarek and WP, work in progress.

[12] It may be described by the distribution based on Gaussian Orthogonal Ensemble of the random matrices.

Chaotic dynamics:

- Classical dynamics of the vacuum Bianchi IX model is chaotic[10]
- What about quantum dynamics[11]?
- Universality conjecture for energy levels distribution:
  - chaotic classical systems are characterized by the Gaussian like distribution describing the 'level repulsion' in quantum theory[12]:

$$P_{GOE}(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right) . \quad (39)$$

  - regular classical systems are characterized by the Poisson like distribution describing 'level clustering' in quantum theory:

$$P_{Poisson}(s) = e^{-s}, \quad (40)$$

  where s is level-spacing.

[10] see, e.g., N. J. Cornish and J. J. Levin, "The Mixmaster universe is chaotic", Phys. Rev. Lett. **78** (1997) 998; "The Mixmaster universe: A Chaotic Farey tale", Phys. Rev. D **55** (1997) 7489

[11] J. Mielczarek and WP, work in progress.

[12] It may be described by the distribution based on Gaussian Orthogonal Ensemble of the random matrices.

Chaotic dynamics (cont):

In real system, chaotic and regular regimes may coexist: the level spacing distribution can be modelled by distribution interpolating between Poisson like and Gaussian like distributions, such as the Brody distribution

$$P_{Brody}(s, \beta) = (\beta + 1) b s^{\beta} \exp\left(-b s^{\beta+1}\right), \tag{41}$$

where

$$b = \left[\Gamma\left(\frac{\beta+2}{\beta+1}\right)\right]^{\beta+1}. \tag{42}$$

Classical Hamiltonian

The action integral for Bianchi type models in Misner's variables:

$$I = \int (p_+ d\beta_+ + p_- d\beta_- - Hd\Omega), \tag{43}$$

where $\beta_\pm,\ p_\pm$, Ω, are independent variables, and $H = H(\Omega, \beta_\pm, p_\pm)$.
An evolution parameter (time) Ω is related to the volume density via

$$\Omega = -\frac{1}{3} \ln \sqrt{g}. \tag{44}$$

Thus, the gravitational system enters the singularity regime when the volume vanishes $\sqrt{g} \to 0$, i.e. $\Omega \to +\infty$.
For the Bianchi IX model we have

$$H^2 = p_+^2 + p_-^2 + e^{-4\Omega}\ V, \tag{45}$$

where the potential reads

$$V = -\frac{4}{3} e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + \frac{2}{3} e^{4\beta_+} (\cosh(4\sqrt{3}\beta_-) - 1) + \frac{1}{3} e^{-8\beta_+}. \tag{46}$$

Classical Hamiltonian

The action integral for Bianchi type models in Misner's variables:

$$I = \int (p_+ d\beta_+ + p_- d\beta_- - H d\Omega), \tag{43}$$

where $\beta_\pm,\ p_\pm$, Ω, are independent variables, and $H = H(\Omega, \beta_\pm, p_\pm)$.
An evolution parameter (time) Ω is related to the volume density via

$$\Omega = -\frac{1}{3} \ln \sqrt{g}. \tag{44}$$

Thus, the gravitational system enters the singularity regime when the volume vanishes $\sqrt{g} \to 0$, i.e. $\Omega \to +\infty$.

For the Bianchi IX model we have

$$H^2 = p_+^2 + p_-^2 + e^{-4\Omega}\ V, \tag{45}$$

where the potential reads

$$V = -\frac{4}{3} e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + \frac{2}{3} e^{4\beta_+} (\cosh(4\sqrt{3}\beta_-) - 1) + \frac{1}{3} e^{-8\beta_+}. \tag{46}$$

Classical Hamiltonian

The action integral for Bianchi type models in Misner's variables:

$$I = \int (p_+ d\beta_+ + p_- d\beta_- - H d\Omega), \tag{43}$$

where $\beta_\pm,\ p_\pm$, Ω, are independent variables, and $H = H(\Omega, \beta_\pm, p_\pm)$.
An evolution parameter (time) Ω is related to the volume density via

$$\Omega = -\frac{1}{3} \ln \sqrt{g}. \tag{44}$$

Thus, the gravitational system enters the singularity regime when the volume vanishes $\sqrt{g} \to 0$, i.e. $\Omega \to +\infty$.
For the Bianchi IX model we have

$$H^2 = p_+^2 + p_-^2 + e^{-4\Omega}\ V, \tag{45}$$

where the potential reads

$$V = -\frac{4}{3} e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + \frac{2}{3} e^{4\beta_+} (\cosh(4\sqrt{3}\beta_-) - 1) + \frac{1}{3} e^{-8\beta_+}. \tag{46}$$

Classical Hamiltonian (cont)

The equipotential lines for the Bianchi IX potential:

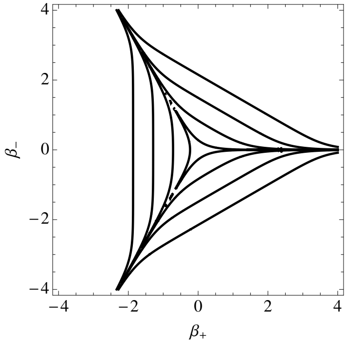

Figure: As time increases, the potential becomes triangle like. It is confining and has the $\mathbb{C}_{3v}$ symmetry.

Quantum Hamiltonian

Making the canonical mapping

$$\Omega \mapsto \tau := e^{-2\Omega}, \quad H \mapsto H_\tau := -\frac{1}{2} e^{2\Omega} H, \tag{47}$$

with unchanged $\beta_\pm$ and $p_\pm$ variables, leads to the Hamiltonian:

$$4H_\tau^2 = \tau^{-2}(p_+^2 + p_-^2) + V - 1, \qquad V = V(\beta_\pm). \tag{48}$$

The singularity occurs at finite time $\tau = 0$.
Quantum operator corresponding to (48) reads

$$4\widehat{H_\tau^2} = -\frac{1}{\tau^2}\Big(\frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta_-^2}\Big) + V(\beta_\pm) - 1. \tag{49}$$

Quantum Hamiltonian

Making the canonical mapping

$$\Omega \mapsto \tau := e^{-2\Omega}, \quad H \mapsto H_\tau := -\frac{1}{2}e^{2\Omega}H, \tag{47}$$

with unchanged $\beta_\pm$ and $p_\pm$ variables, leads to the Hamiltonian:

$$4H_\tau^2 = \tau^{-2}(p_+^2 + p_-^2) + V - 1, \qquad V = V(\beta_\pm). \tag{48}$$

The singularity occurs at finite time $\tau = 0$.
Quantum operator corresponding to (48) reads

$$4\widehat{H_\tau^2} = -\frac{1}{\tau^2}\left(\frac{\partial^2}{\partial\beta_+^2} + \frac{\partial^2}{\partial\beta_-^2}\right) + V(\beta_\pm) - 1. \tag{49}$$

Eigenvalue problem for Hamiltonian

For statistical analysis we should solve the eigen problem:

$$\widehat{H_\tau} f_k(\tau, \beta_+, \beta_-) = e_k(\tau) f_k(\tau, \beta_+, \beta_-), \tag{50}$$

where $k \in \mathbb{Z}$, $e_k(\tau) \in \mathbb{R}$, and $\{f_k\}_{k\in\mathbb{Z}}$ can be used to determine an orthonormal basis in the subspace D_τ, where $D_\tau \subset \mathcal{H}_\tau := L^2(S_\tau, d\mu)$, chosen in such a way that $\hat{H}_\tau$ is essentially self-adjoint on D_τ. The subset S_τ is defined as:

$$S_\tau := \{(\beta_-, \beta_+) \in \mathbb{R}^2 \mid \tau^{-2}(p_+^2 + p_-^2) + V(\beta_\pm) - 1 > 0, \ \forall\, (p_-, p_+) \in \mathbb{R}^2\}, \tag{51}$$

where $0 < \tau < \tau_0$ defines the monotonicity interval of time.

Triangle potential well approximation:

- The eigen problem for the $\widehat{H_\tau}$ is mathematically equivalent to solving the Schrödinger equation for a particle in two dimensional potential well.
- The difficulty in solving this equation is due to complicated form of the potential, which would require sophisticated numerical techniques.
- Near the singularity, the potential can be approximated by the hard walls equilateral triangle potential.

Triangle potential well approximation:

- The eigen problem for the $\widehat{H_\tau}$ is mathematically equivalent to solving the Schrödinger equation for a particle in two dimensional potential well.
- The difficulty in solving this equation is due to complicated form of the potential, which would require sophisticated numerical techniques.
- Near the singularity, the potential can be approximated by the hard walls equilateral triangle potential.

Triangle potential well approximation:

- The eigen problem for the $\widehat{H_\tau}$ is mathematically equivalent to solving the Schrödinger equation for a particle in two dimensional potential well.
- The difficulty in solving this equation is due to complicated form of the potential, which would require sophisticated numerical techniques.
- Near the singularity, the potential can be approximated by the hard walls equilateral triangle potential.

Triangle potential well approximation (cont)

The eigen problem of a particle in equilateral triangle was solved analytically[13]. The eigenvalues of $\widehat{H^2_\tau}$ are:

$$e^2_{q,p} = (p^2 + pq + q^2)E_0, \tag{52}$$

where $E_0 > 0$ is a constant, and

$$q = \begin{cases} 0, 1, 2, \ldots, \\ 1, 2, 3, \ldots, \\ \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \ldots, \end{cases} \tag{53}$$

and where

$$p = q + 1, q + 2, \ldots \tag{54}$$

Taking the spectral square root of the operator $\widehat{H^2_\tau}$, one gets the spectrum of $\widehat{H}_\tau$:

$$e_{q,p}(\tau) \sim \sqrt{p^2 + pq + q^2}. \tag{55}$$

[13] Wai-Kee Li, S.M. Blinder, *Solution of the Schrödinger equation for a particle in a equilateral triangle*, J. Math. Phys. **26**, 2784 (1985)

Triangle potential well approximation (cont)

The eigen problem of a particle in equilateral triangle was solved analytically[13]. The eigenvalues of $\widehat{H_\tau^2}$ are:

$$e_{q,p}^2 = (p^2 + pq + q^2)E_0, \tag{52}$$

where $E_0 > 0$ is a constant, and

$$q = \begin{cases} 0, 1, 2, \ldots, \\ 1, 2, 3, \ldots, \\ \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \ldots, \end{cases} \tag{53}$$

and where

$$p = q + 1, q + 2, \ldots \tag{54}$$

Taking the spectral square root of the operator $\widehat{H_\tau^2}$, one gets the spectrum of $\widehat{H}_\tau$:

$$e_{q,p}(\tau) \sim \sqrt{p^2 + pq + q^2}. \tag{55}$$

[13]Wai-Kee Li, S.M. Blinder, *Solution of the Schrödinger equation for a particle in a equilateral triangle*, J. Math. Phys. **26**, 2784 (1985)

Level-spacing distribution for the Bianchi IX model:

Computational procedure:

- Bianchi IX potential is approximated by the hard wall equilateral triangle potential with $e_{q,p}(\tau) \sim \sqrt{p^2 + pq + q^2}$.
- quantum numbers necessary to parameterize levels of different energy are chosen as follows

$$q = \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \ldots, \quad p = q + 1, q + 2, \ldots \tag{56}$$

- corresponding set of eigenvalues E_i is such that $E_1 < E_2 < E_3 < E_4 < \cdots < E_N$
- set of level-spacings $\Delta_i := E_{i+i} - E_i$,
- $s_i := \Delta_i / \bar{\Delta}$ are time independent

Level-spacing distribution (cont):

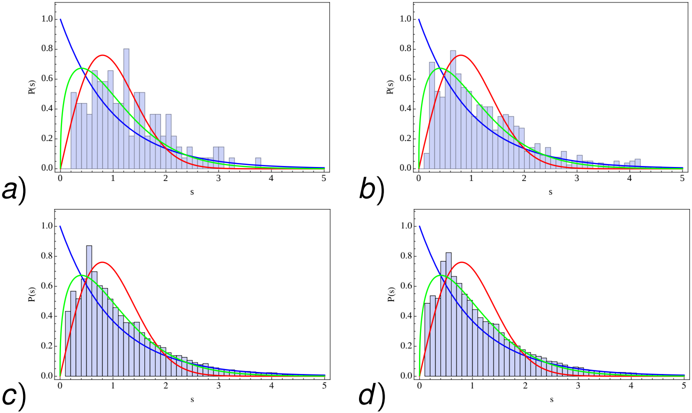

Figure: Level-spacing distributions for different number of levels taken into account: a) 310, b) 1742, c) 53431, d) 142887. The blue line corresponds to the Poisson like distribution. The red line is the Gaussian like distribution. The green line is the Brody distribution with the parameter $\beta = 0.35$.

Summary:

- Quantum dynamics of a particle in hard wall triangle is satisfactory described by Brody's distribution.
- Examination of fluctuations of the distribution by unfolding procedure may bring some new information concerning the chaoticity of the distribution.
- The statistics is time independent as the variable s is time independent.
- This toy model cannot be used to see what happens at the singularity.
- Open question: What is the quantum dynamics of the Bianchi IX with realistic potential?

Summary:

- Quantum dynamics of a particle in hard wall triangle is satisfactory described by Brody's distribution.
- Examination of fluctuations of the distribution by unfolding procedure may bring some new information concerning the chaoticity of the distribution.
- The statistics is time independent as the variable s is time independent.
- This toy model cannot be used to see what happens at the singularity.
- Open question: What is the quantum dynamics of the Bianchi IX with realistic potential?

Summary:

- Quantum dynamics of a particle in hard wall triangle is satisfactory described by Brody's distribution.
- Examination of fluctuations of the distribution by unfolding procedure may bring some new information concerning the chaoticity of the distribution.
- The statistics is time independent as the variable s is time independent.
- This toy model cannot be used to see what happens at the singularity.
- Open question: What is the quantum dynamics of the Bianchi IX with realistic potential?

Summary:

- Quantum dynamics of a particle in hard wall triangle is satisfactory described by Brody's distribution.
- Examination of fluctuations of the distribution by unfolding procedure may bring some new information concerning the chaoticity of the distribution.
- The statistics is time independent as the variable s is time independent.
- This toy model cannot be used to see what happens at the singularity.
- Open question: What is the quantum dynamics of the Bianchi IX with realistic potential?

Summary:

- Quantum dynamics of a particle in hard wall triangle is satisfactory described by Brody's distribution.
- Examination of fluctuations of the distribution by unfolding procedure may bring some new information concerning the chaoticity of the distribution.
- The statistics is time independent as the variable s is time independent.
- This toy model cannot be used to see what happens at the singularity.
- Open question: What is the quantum dynamics of the Bianchi IX with realistic potential?

Next steps for the Bianchi IX model:

- Studying classical evolution near the singularity by dynamical systems method to find suitable canonical formulation convenient for quantization.
- Examination of statistics of energy spectrum
  - vacuum or perfect fluid - classical chaos does occur,
  - massless scalar field - classical chaos may be absent,
  - massless vector field - not examined yet.
- Rigorous quantization of dynamics: evolution of BIX towards the cosmological singularity can be considered to be a sequence of transitions from one Kasner epoch to another via vacuum BII evolution.[14]
- Making predictions for primordial gravitational waves.

Successful quantization of the Bianchi IX model may enable quantization of the BKL scenario.

---

[14] H. Bergeron, O. Hrycyna, P. Małkiewicz, and WP, 'Quantum theory of the Bianchi II model', Phys. Rev. D, in press.

Next steps for the Bianchi IX model:

- Studying classical evolution near the singularity by dynamical systems method to find suitable canonical formulation convenient for quantization.
- Examination of statistics of energy spectrum
  - vacuum or perfect fluid - classical chaos does occur,
  - massless scalar field - classical chaos may be absent,
  - massless vector field - not examined yet.
- Rigorous quantization of dynamics: evolution of BIX towards the cosmological singularity can be considered to be a sequence of transitions from one Kasner epoch to another via vacuum BII evolution.[14]
- Making predictions for primordial gravitational waves.

Successful quantization of the Bianchi IX model may enable quantization of the BKL scenario.

[14] H. Bergeron, O. Hrycyna, P. Małkiewicz, and WP, 'Quantum theory of the Bianchi II model', Phys. Rev. D, in press.

Next steps for the Bianchi IX model:

- Studying classical evolution near the singularity by dynamical systems method to find suitable canonical formulation convenient for quantization.
- Examination of statistics of energy spectrum
  - vacuum or perfect fluid - classical chaos does occur,
  - massless scalar field - classical chaos may be absent,
  - massless vector field - not examined yet.
- Rigorous quantization of dynamics: evolution of BIX towards the cosmological singularity can be considered to be a sequence of transitions from one Kasner epoch to another via vacuum BII evolution.[14]
- Making predictions for primordial gravitational waves.

Successful quantization of the Bianchi IX model may enable quantization of the BKL scenario.

[14] H. Bergeron, O. Hrycyna, P. Małkiewicz, and WP, 'Quantum theory of the Bianchi II model', Phys. Rev. D, in press.

Next steps for the Bianchi IX model:

- Studying classical evolution near the singularity by dynamical systems method to find suitable canonical formulation convenient for quantization.
- Examination of statistics of energy spectrum
  - vacuum or perfect fluid - classical chaos does occur,
  - massless scalar field - classical chaos may be absent,
  - massless vector field - not examined yet.
- Rigorous quantization of dynamics: evolution of BIX towards the cosmological singularity can be considered to be a sequence of transitions from one Kasner epoch to another via vacuum BII evolution.[14]
- Making predictions for primordial gravitational waves.

Successful quantization of the Bianchi IX model may enable quantization of the BKL scenario.

[14] H. Bergeron, O. Hrycyna, P. Małkiewicz, and WP, 'Quantum theory of the Bianchi II model', Phys. Rev. D, in press.

Next steps for the Bianchi IX model:

- Studying classical evolution near the singularity by dynamical systems method to find suitable canonical formulation convenient for quantization.
- Examination of statistics of energy spectrum
  - vacuum or perfect fluid - classical chaos does occur,
  - massless scalar field - classical chaos may be absent,
  - massless vector field - not examined yet.
- Rigorous quantization of dynamics: evolution of BIX towards the cosmological singularity can be considered to be a sequence of transitions from one Kasner epoch to another via vacuum BII evolution.[14]
- Making predictions for primordial gravitational waves.

Successful quantization of the Bianchi IX model may enable quantization of the BKL scenario.

[14] H. Bergeron, O. Hrycyna, P. Małkiewicz, and WP, 'Quantum theory of the Bianchi II model', Phys. Rev. D, in press.

Next steps for the Bianchi IX model:

- Studying classical evolution near the singularity by dynamical systems method to find suitable canonical formulation convenient for quantization.
- Examination of statistics of energy spectrum
  - vacuum or perfect fluid - classical chaos does occur,
  - massless scalar field - classical chaos may be absent,
  - massless vector field - not examined yet.
- Rigorous quantization of dynamics: evolution of BIX towards the cosmological singularity can be considered to be a sequence of transitions from one Kasner epoch to another via vacuum BII evolution.[14]
- Making predictions for primordial gravitational waves.

Successful quantization of the Bianchi IX model may enable quantization of the BKL scenario.

[14] H. Bergeron, O. Hrycyna, P. Małkiewicz, and WP, 'Quantum theory of the Bianchi II model', Phys. Rev. D, in press.

Thank you!