[P4] [*Computational Hardness of the general Ising ground state problem*] Recall the *general Ising ground state problem*: Given an Ising model – i.e. a weighted graph with vertices ("spins") $V = \{\sigma_1, \ldots, \sigma_n\}$ and a weighted adjacency matrix $J \in M_{n\times n}(\mathbb{R})$ that encodes the interaction strength with Hamiltonian

$$H(\sigma_1, \ldots, \sigma_n) = -\sum_{i,j=1}^{n} J_{i,j}\sigma_i\sigma_j, \tag{1}$$

– and a number $k \in \mathbb{R}$, decide whether there is an assignment $\sigma_1^\sharp, \ldots, \sigma_n^\sharp$ with $\sigma_i^\sharp \in \{-1, 1\}$ such that

$$H\left(\sigma_1^\sharp, \ldots, \sigma_n^\sharp\right) \leq k. \tag{2}$$

The goal of this exercise is to show that such a problem is in general NP-complete. We suggest to prove NP-hardness via establishing the following sequence of reductions:

$$\text{SAT} \leq_p \text{3-SAT} \leq_p \text{NAE 3-SAT} \leq_p \text{MAX-CUT} \leq_p \text{weighted MAX-CUT} \leq_P \text{Ising}. \tag{3}$$

Since the Cook-Levin Theorem asserts that SAT is NP hard, such a reduction establishes NP-hardness of Ising and has the added benefit of visiting some famous NP-complete problems along the way.

(1) *SAT $\leq_p$ 3-SAT:* Similar to SAT, 3-SAT asks whether there is a satisfiable assignment for a boolean formula that is in conjunctive normal form where each clause is limited to at most three literals, e.g. $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_4 \vee x_5 \vee \neg x_6) \cdots$. We refer to boolean formulas having such a structure as 3-CNF's. Show that 3-SAT is NP-hard by reducing any instance of SAT to it with a polynomial overhead. (2 P.)

(2) *3-SAT $\leq_p$ NAE 3-SAT:* Consider a boolean 3-CNF formula. NAE-3-SAT ("not all equal") asks whether there is an assignment such that each clause contains at least one true literal and at least one false literal. Show that NAE 3-SAT is NP hard by reducing 3-SAT to it.

Hint: Reduce 3-SAT to NAE 4-SAT first and in a second step break up the resulting clauses into 3-CNFs. (2 P.)

(3) *NAE 3-SAT $\leq_p$ MAX-CUT:* Given an undirected graph $G = (V, E)$ with vertices V, a cut is a subset $S \subseteq V$. The *size* of this cut is defined to be the number of edges with one end in S and the other one in S^c. Given $k \in \mathbb{N}$, MAX-CUT asks whether G admits a cut of size greater than or equal to k. Show that MAX-CUT is NP hard by reducing NAE 3-SAT to it.

Hint: if the NAE 3-SAT formula has k clauses, represent each variable x_i with $3k$ vertices labeled x_i and another $3k$ ones labeled $\neg x_i$. Introduce $(3k)^2$ edges that connect all the x_i-vertices with all the $\neg - x_i$ ones. Then represent each clause with a triangle connecting an appropriate triple of vertices. Figure out how large the cut that corresponds to a solution of the NAE 3-SAT formula is and prove that a cut of this size exists if and only if the formula is satisfiable. (3 P.)

(4) *MAX-CUT $\leq_p$ weighted MAX-CUT:* Weighted MAX-CUT is a generalization of MAX-CUT for weighted, undirected graphs. Accordingly, the size of a cut is defined to

be the sum of the weights associated to the edges that connect S and S^c. Justify that weighted MAX-CUT is NP hard by arguing that it is at least as difficult as MAX-CUT. (1 P.)

(5) *weighted MAX-CUT $\leq_p$ Ising:* Prove that the generalized Ising problem introduced above is NP-hard by reducing weighted MAX-CUT to it. Complete the argument by showing that Ising is itself in NP which renders the problem NP-complete.
Hint: Construct a one-to-one relation between cuts of a graph and Ising-configurations $\sigma_1, \ldots, \sigma_n$. This correspondence should allow you to conclude that minimizing the Ising model's energy also establishes a cut of maximal size. (2 P.)