[P9] (Coupling spins via bus qubit in ion-based quantum computer). In the lecture we saw that the following Hamiltonians can be realized in the trapped-ion architecture (the subscripts "b" and "r" allude to a certain "red" or "blue" detuning):

$$H_{\mathrm{b}}(\phi) = S_+ a^\dagger e^{i\phi} + S_- a\, e^{-i\phi}, \qquad H_{\mathrm{r}}(\phi) = S_+ a\, e^{i\phi} + S_- a^\dagger e^{-i\phi},$$

where ϕ is a tunable phase, S_- is the ladder operator for the spin-1/2 degree of freedom of a given ion ($S_-|0\rangle = 0, S_-|1\rangle = |0\rangle$), S_+ is its adjoint, and $a : |n\rangle \mapsto \sqrt{n}|n-1\rangle$ is the usual annihilation operator acting on the center-of-mass motional mode. A basis of the joint system formed by one ion and the bus is given by

$$\{|m, n\rangle \,|\, m \in \{0, 1\}, n \in \mathbb{N}_0\},$$

where m encodes direction of the spin and n the number of phonons in the bus mode. On the bus, we will use $|m = 0\rangle$ to encode a logical "0" and $|m = 1\rangle$ for "1". States with more than one phonon do not carry a meaning in our scheme, and we must therefore take care not to excite them. With this in mind, we refer to the (four-dimensional) subspace $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ of the (infinite-dimensional) physical Hilbert space of the joint system as the *computational subspace*.

(1) First, we verify that we can move information from the spin onto the bus. Assume that the bus is initially in the ground state $|0\rangle$, whereas the spin is in an arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Let $T : |0\rangle \mapsto |0\rangle, |1\rangle \mapsto -i|1\rangle$ be a unitary operation acting only on the spin (we will work out how to realize such transformations in the lecture). Show that

$$e^{i\frac{\pi}{2} H_{\mathrm{r}}(0)} (T \otimes \mathbb{1})(|\psi\rangle \otimes |0\rangle) = |0\rangle \otimes |\psi\rangle.$$

(2) Unfortunately, H_{b} couples the computational subspace to the state $|1, 2\rangle$ which features two phonons. Here, we will show that one can nevertheless use H_{b} to perform transformations within the computational space. Indeed, show that in the basis $\{|10\rangle, |00\rangle, |11\rangle, |01\rangle, |12\rangle\}$ (in that order!), the Hamiltonian becomes block-diagonal. Use this representation to prove that $e^{itH_{\mathrm{b}}(\phi)}$ does preserve the computational subspace if t is a multiple of $\frac{\pi}{\sqrt{2}}$. What is more, show that for every t and every ϕ_1, ϕ_2, the unitary

$$U = e^{itH_{\mathrm{b}}(\phi_1)}\, e^{i\frac{\pi}{\sqrt{2}} H_{\mathrm{b}}(\phi_2)}\, e^{-itH_{\mathrm{b}}(\phi_1)}$$

preserves the computational subspace (hint: in the block representation, no calculations are necessary for this).

(3) Show (using a computer algebra system if necessary) that choosing $\phi_1 = -\pi/2, \phi_2 = 0, t = \pi/4$ in the definition for U above, one obtains a diagonal matrix. Result: With respect to the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, one should get

$$W = \begin{pmatrix} e^{-i\pi/\sqrt{2}} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\pi/\sqrt{2}} \end{pmatrix}.$$

Let $S(\gamma)$ be the unitary that maps $|0\rangle \to |0\rangle, |1\rangle \to e^{i\gamma}|1\rangle$. Find phases $\gamma_1, \gamma_2, \gamma_3$ such that $e^{i\gamma_1}\big(S(\gamma_2) \otimes S(\gamma_3)\big)W$ is the controlled-Z gate.

(4) How does all this allow us to perform a controlled-Z gate between two spins in the ion trap? (10 P.)