Quantitative Entanglement Measures

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Axiomatic approach

To measure entanglement a simple binary test is not sufficient. A quantitative measure should yield non-integer results. Any entanglement measure E should [1]...

1. *...vanish on separable states.*

2. ...be *invariant w.r.t. local unitary operations* (e.g. rotation of basis).

3. ...fulfill *monotonicity unter LOCC* [1] Λ, i.e.

$$E(\Lambda(\rho)) \leq E(\rho)$$

   3a. ...not increase under LOCC *on average* (this is optional, but useful):
   For an ensemble $\{p_i, \rho_i\}$ obtained from Λ,

$$\sum_i p_i E(\rho_i) \leq E(\rho).$$

4. *Other possible postulates* (optional, but common):

   4a. *Normalization*: Allows to count *e-bits* on maximally entangled states:

$$E((\Phi_2^+)^{\otimes n}) = n$$

   4b. *asymptotic continuity*:

$$\|\rho - \sigma\| \to 0 \Rightarrow |E(\rho) - E(\sigma)| \to 0$$

   If 4a. & 4b. are both fulfilled, an ent. measure will be unique for pure states.

   4c. *Convexity*: cf. *convex roof measures* below.

Examples for entanglement measures[1]

Convex roof measures

For a given ρ, let $S = \{\{p_i, |\Psi_i\rangle\}\}$ s.t.

$$\rho = \sum_i p_i |\Psi_i\rangle \langle\Psi_i| \Rightarrow E(\rho) = \inf_S \sum_i p_i E(\Psi_i)$$

The infimum may be attained on an *optimal ensemble.* Use e.g. von Neumann entropy $E(\Psi) = -tr(\rho \, log_2 \rho)$

Distance-based measures

Closeness to a set S of separable states implies less entanglement:

$$E_{D,S}(\rho) = \inf_{\sigma \in S} D(\rho, \sigma)$$

Ent. distillation & dilution[2]

View ent. as a quantifiable physical resource, e.g. with Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ as standard unit of measure: If m copies of $|\Psi\rangle$ may be turned into n copies of $|\Phi^+\rangle$, define n/m as the *distillable ent.* of $|\Psi\rangle$. n/m = 1 for maximally entengled states.

$$|\Psi\rangle^{\otimes m} \underset{dilution,\, m<n}{\overset{distillation,\, n<m}{\rightleftarrows}} |\Phi^+\rangle^{\otimes n}$$

A simple dilution protocol

Alice and Bob share a number of q-bits and want to transform them by LOCC into as many copies of state $|\Psi\rangle$ with Schmidt decomposition $|\Psi\rangle = \sum_x \sqrt{p(x)}\, |x_A\rangle\, |x_B\rangle$ as possible, with a fidelity defined by ϵ-typicality:

$$|\Psi\rangle^{\otimes m} \approx |\varphi_m\rangle = \sum_{\substack{x\ \epsilon- \\ typical}} \sqrt{p(x_1)...p(x_m)}\, |x_{1A}...x_{mA}\rangle\, |x_{1B}...x_{mB}\rangle$$

In this way, there are at most $2^{m(S(\rho_\Psi)+\epsilon}$ terms in $|\varphi_m\rangle$ (S being the von Neumann entropy). Thus, from (asymptotically) $S(\rho_\Psi)$ Bell states, a single copy of Ψ may be obtained. By reversing the process, Ψ may (asymptotically) be distilled into $S(\rho_\Psi)$ Bell states.
With this protocol, the von Neumann entropy does not just provide an entanglement measure, i.e. give some arbitrary number, but carries actual meaning for an operational process.

References

[1] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. Quantum entanglement. *Rev. Mod. Phys.*, 81:865–942, Jun 2009.

[2] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition.* Cambridge University Press, New York, NY, USA, 10th edition, 2011.

[1] Local Operations and Classical Calls: After sharing an initial state, no quantum communication between Alice and Bob is allowed. They may perform *arbitrary* operations on their local systems and use classical communication only.