CLASSIFYING TOPOLOGICAL PHASES WITH PROJECTIVE SYMMETRIES

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Abstract: for a long time it was believed that all kind of phase transitions could be described by Landau Symmetry-Breaking. The properties of the system are described by a local order parameter and in order to change the phase, a symmetry has to break. The discovery of the Quantum Hall Effect makes it necessary to develop the new theory of topological phases. The Classification turns out to be a hard task and can be simplified by projective symmetries.

Topological Phases: Two Hamiltonians $\mathcal{H}_a^s, \mathcal{H}_b^s$ with symmetry group G_s belong to the same topological class (phase) if $\exists \mathcal{H}_\lambda^S$ homotopy connecting both Hamiltonians conserving the symmetry as well as the bulk gap.

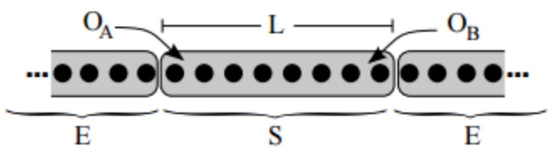
Altland-Zirnbauer Classification: All phases of free fermionic Hamiltonians with time reversal (T) - , sublattice (S) -, particle-hole (C) - symmetries, are fully classified by A. Altland and M. Zirnbauer.

	Symmetry			Dimension							
AZ	T	C	S	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

These symmetries are the most robust against perturbation an therefore are the most relevant in nature.

Interacting Hamiltonians: In presence of interactions (=quartic terms in the Hamiltonian) one drops out of AZ-Classification and going via Homotopies is a cruel task. A better way in 1D is the commutation relation of projective representations of the symmetry operators. These are the same within a topological phase and are called topological invariants.

Projective Symmetry Representation: We consider a gapped 1D Chain with finite correlation length ξ, bipartitioned into system S and environment E.



When the Action of an operator on S $\hat{O}_S$ is sufficiently far away from its action on E $\hat{O}_E$, that the product expectation value decouples. Lets consider the ground state in the Schmidt decomposition $\|\psi\rangle = \sum \alpha_i |\phi_i\rangle \otimes |\chi_i\rangle$ and a projector onto one schmidtstate as $\hat{O}_E = |\chi_\gamma\rangle\langle\chi_{\gamma'}|$.

$$\alpha_\gamma^2 \langle \hat{O}_s \rangle_{\phi_\gamma} = \langle \hat{O}_S \hat{O}_E \rangle_\psi \approx \langle \hat{O}_S \rangle_\psi \langle \hat{O}_E \rangle_\psi = \alpha_\gamma^2 \langle \hat{O}_S \rangle_\psi \ , \quad \gamma = \gamma'$$
$$\alpha_\gamma \alpha_{\gamma'} \langle \phi_\gamma | \hat{O}_s | \phi_{\gamma'} \rangle = \langle \hat{O}_S \hat{O}_E \rangle_\psi \approx \langle \hat{O}_S \rangle_\psi \langle \hat{O}_E \rangle_\psi = 0 \ , \quad \gamma \neq \gamma'$$

We see that different entanglement eigenstates are not connected and behave all like the ground state. The action of an arbitrary symmetry operator on the system is then determined by its action near the edges A and B. We name

$$\hat{O}_S = \hat{O}_A \hat{O}_B \tag{1}$$

a projective representation. The (global) topological invariant $\mu = 0, \pi$, which determines the commutation relation $\hat{O}_A \hat{O}_B = e^{i\mu} \hat{O}_B \hat{O}_A$ can be used to classify the occuring phases.

Example: As an example we consider an interacting 1D fermionic chain in which 2 symmetries are maintained by interactions: parity conservation ($\#_{fermions}\ \mathbf{mod}\ 2$) and time reversal. From the condition that the parity operator $\hat{Q} = \alpha \hat{Q}_A \hat{Q}_B$ squares to identity

$$\mathbb{1} = \hat{Q}^2 = \alpha^2 \left(\hat{Q}_A [\hat{Q}_B, \hat{Q}_A]_\mp \hat{Q}_B \pm \mathbb{1} \right) \ , \tag{2}$$

one obtains, that either $\hat{Q}_A, \hat{Q}_B$ commute with $\alpha^2 = 1$ and therefore the first topological invariant is $\phi = \pi$ or they anti commute with $\alpha^2 = -1$ and therefore $\phi = 0$.
We would like to recall the Kitaev chain in a non trivial topological phase in Majorana representation $\mathcal{H}_{\text{chain}} = -i \sum \gamma_{j,1} \gamma_{j+1,2}$ [2]. Then we can set $\hat{Q}_A = \gamma_{0,1}$, $\hat{Q}_B = \gamma_{N,2}$ and see that they act only near the edges of our system and this phase corresponds to a $\phi = \pi$ phase.
The second symmetry, time reversal, is represented by an *anti unitary* operator. Its projective representation is given by $T = \hat{U}_a \hat{U}_B \mathcal{K}$, where $\mathcal{K}$ denotes the complex conjugation operator. The second topological invariant is defined by the commutation relation $\hat{U}_A \hat{U}_B = e^{i\kappa} \hat{U}_B \hat{U}_A$. Our third invariant follows from the condition that

$$\mathbb{1} = T^2 = e^{i\kappa} \hat{U}_A \hat{U}_A^* \hat{U}_B \hat{U}_B^* \ , \tag{3}$$

which implies that

$$\hat{U}_A \hat{U}_A^* = \hat{U}_B \hat{U}_B^* = e^{i\tau} \mathbb{1} \tag{4}$$

All in all, one has three binary topological invariants, which implies that there are 8 different combinations. A 1D interacting spin system maintaining these symmetries has therefore only 8 distinct topological phases. When two such systems are combined together, their phases follow an additive group structure isomorphic to the $\mathbb{Z}_8$. If we additionally require translation symmetry to be maintained by the interactions the group structure becomes isomorphic to the $\mathbb{Z}_{16}$.

References:

[1] A.M. Turner, F. Pollmann, E. Berg, "Topological phases of one-dimensional fermions: An entanglement point of view" *Phys. Rev. B 83*, 075102, 2011.

[2] S. Matern, F. Chen "The Kitaev chain: theoretical model and experiments" *Knot Seminar*: seminar talk 8, `http://www.thp.uni-koeln.de/trebst/Lectures/2014-KnotSeminar.html`, Jul 2014.