Topological states: Kitaev's toric code

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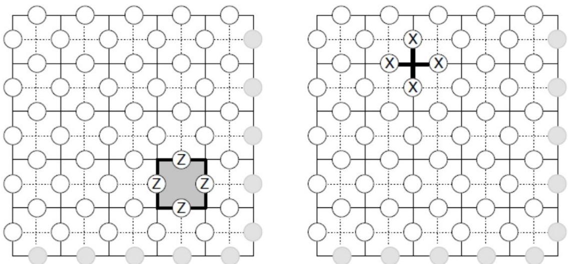
Abstract

Kitaev's toric code is an exactly solvable spin $\frac{1}{2}$ model on a $L \times L$ square lattice with periodic boundary conditions. The latter makes the square topologically equivalent to a torus. In order to generate a physical system that can be used in quantum information theory we need a protected area in Hilbert space. Such a degenerate ground state can be provided by topology. We then end up having so-called topological phases, i.e. gapped phases with a protected ground state degeneracy dependent on the topology of the underlying manifold.

Stabilizer formalism. We describe the physical system by positioning a qubit (or spin $\frac{1}{2}$ particle) on every edge of the lattice and use Pauli matrices X and Z. We then introduce plaquette B_p and vertex A_s operators as

$$B_p \equiv \prod_{j \in \partial p} Z_j \qquad A_s \equiv \prod_{j \in s} X_j$$



One finds that these operators fulfill the commutation relations

$$[B_p, B_{p'}] = [A_s, A_{s'}] = [A_s, B_p] = 0$$

Within the stabilizer formalism we can define so-called *encoded logical operators* $\hat{X}_1$, $\hat{Z}_1$, $\hat{X}_2$ and $\hat{Z}_2$. These have to commute with all stabilizer elements (which are generated by the B_p's and A_s's) and must not be elements of the stabilizer group themselves. One finds that these conditions are met by operators which wrap around the torus (so they are non-contractible) and are either made up of only X or Z Pauli operators (so have four of them).

These logical operators are not uniquely though, but rather define each an equivalence class, since they can be "deformed" by the application of e.g. plaquette operators, which leave a code word state invariant, e.g. $\hat{Z}_1 B_p |\psi\rangle = \hat{Z}_1 |\psi\rangle$.

Quasi particle interpretation. If the outcome of a measurement results in an eigenvalue $+1$ we interpret this as "no" particle and in the case of an eigenvalue -1 as an electric charge e (for Z operators) or a magnetic charge m (for X operators). Or in other words: there is a charge (e or m) if an operator (e.g. a string) anticommutes with the Hamiltonian which we later define. Loops (which do not wrap around the torus) create, move and annihilate quasi particle and can be regarded as homotopy equivalent to the identity.

Many body picture. The introduced machinery can also be considered from the point of view of many body physics by firstly introducing the Hamiltonian

$$\mathcal{H}_T = -J_e \sum_s A_s - J_m \sum_p B_p$$

Let s_j denote the eigenvalue of Z_j, then we find that the ground state (in the Z basis) is given by

$$|\psi\rangle = \sum_s c_s |s\rangle$$

with the restriction that all plaquettes are *vortex-free*, i.e. there are no quasi-particles/excitations and all c_s are equal due to the fact that spin flips generated by A_s have to be ergodic.

Excited states can be constructed from the ground state $|\psi\rangle$ by applying a so-called electric charge operator W_l^e (for Z's) which e.g. for a path on the lattice is the product of all Z_j of which the path l consists, i.e. $W_l^e \equiv \prod_{j \in l} Z_j$ and the corresponding excited state can be written as

$$|\psi_l\rangle = W_l^e |\psi\rangle$$

In the context of quasi particles introduced above this excited states corresponds to the creation of two electric charges e which raise the system's energy by $2J_e$.

Braiding. Due to the above commutation relations e and m each obey a bosonic statistic (so the wavefunction remains invariant under the exchange of two particles). But their mutual statistic shows a different behavior. If we exchange an e and a m charge *twice* the wavefunction picks up a minus sign. Particles with such a statistic are called Abelian anyons.

References

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