Quantum Information Theory

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Exercise sheet 1 **Due: April, 18 at 12:00**

1 Basics

Two of the Pauli matrices we are going to use a lot are

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

And the qubits will be often written in terms of the computational basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

or the superposition basis

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

a) Check that the superposition basis $\{|+\rangle, |-\rangle\}$ is indeed an orthonormal basis.

b) Check that $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$, $X|0\rangle = |1\rangle$, and $X|1\rangle = |0\rangle$.

c) Check that in the superposition basis the roles of the Z and X matrices are reversed: $X|+\rangle = |+\rangle$, $X|-\rangle = -|-\rangle$, $Z|+\rangle = |-\rangle$, and $Z|-\rangle = |+\rangle$.

d) There exists a unique matrix H such that $H|0\rangle = |+\rangle$ and $H|1\rangle = |-\rangle$. Write down H as a 2×2 matrix. Check that $H|+\rangle = |0\rangle$ and that $H|-\rangle = |1\rangle$. Why is this the case?

2 Tensor products

a) Check that $(Z \otimes \mathbb{1})(|\psi\rangle|\varphi\rangle) = (Z|\psi\rangle)|\varphi\rangle$ and $(\mathbb{1} \otimes Z)(|\psi\rangle|\varphi\rangle) = |\psi\rangle(Z|\varphi\rangle)$. These operators are often referred to as $Z^{(1)} := Z \otimes \mathbb{1}$ and $Z^{(2)} := \mathbb{1} \otimes Z$.

b) Show that

$$(H|0\rangle)^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{i_1,\dots,i_n=0,1} |i_1\rangle|i_2\rangle\dots|i_n\rangle,$$

and write down the result as column vector for $n = 2$ and $n = 3$.

c) Check that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

d) Show that if some matrices ρ, σ are positive (have eigenvalues ≥ 0) then their tensor prodcut $\rho \otimes \sigma$ is also positive.

e) Show that for any matrices A, B we have that $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$.

f) Show that if ρ, σ are density matrices then their tensor product $\rho \otimes \sigma$ is also a density matrix.

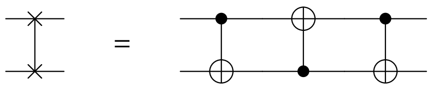
3 Magic state quantum computing

a) The action of the controlled-NOT (CNOT) gate on the computational basis can be written as $|x,y\rangle \mapsto |x, x \oplus y\rangle$, where $x, y \in \{0,1\}$ and $\oplus$ is addition modulo 2. Explicitly, that is

$$|00\rangle \mapsto |00\rangle\,, \quad |01\rangle \mapsto |01\rangle\,, \quad |10\rangle \mapsto |11\rangle\,, \quad |11\rangle \mapsto |10\rangle\,.$$

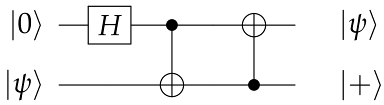
The SWAP gate acts by exchanging states as $|\psi\rangle \otimes |\varphi\rangle \mapsto |\varphi\rangle \otimes |\psi\rangle$. The CNOT and SWAP are represented in a quantum circuit by the symbols

Show that applying a SWAP gate is equivalent to applying three CNOTs, that is, that

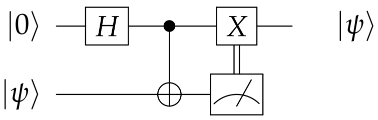


Note that the second CNOT is reversed, meaning that the controlling qubit is now the second one.

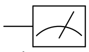
b) Consider the following circuit which swaps a state $|\psi\rangle$ with $|+\rangle = H|0\rangle$:



Show that if we do not care about the final $|+\rangle$ state, the swap can also be realised by exchanging the second CNOT for a Z-measurement on the second qubit and a conditional X-correction on the first one:



(1)

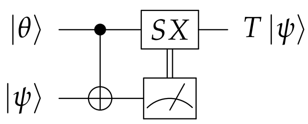
Here,  is the symbol for a measurement in the computational basis and $\|$ means that the X gate is applied if the measurement yields 1 and otherwise not.

c) Finally, we want to show that we can effectively perform certain gates *without actually applying them*. The only ingredient that we need are special states. This procedure is called *state injection* and is the basis for the popular *magic state model of quantum computing*.

The S and T gates are rotations around the Z axis, given by the matrices

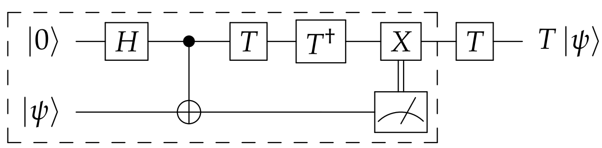
$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

Show that we can effectively apply a T gate on a state $|\psi\rangle$ given the state $|\theta\rangle := T|+\rangle$ as follows:



(2)

Hint: Instead of calculating by brute force, you could note that the circuit in equation (1) is equivalent to the part inside the dashed box of the following circuit:



(3)

Next, show that the whole circuit in equation (3) is equivalent to the one in equation (2) by computing $[T \otimes \mathbb{1}, CNOT]$ and $TXT^\dagger$.