Quantum Information Theory

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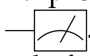
Exercise sheet 2 Due: 2018.05.02 at 12:00

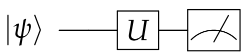
1 Measurements

A von Neumann measurement of a qubit $|\psi\rangle$ with results $\heartsuit$ and $\spadesuit$ is described by two projectors $\{\Pi_\heartsuit, \Pi_\spadesuit\}$ such that $\Pi_\heartsuit + \Pi_\spadesuit = \mathbb{1}$. Result $\heartsuit$ happens with probability $p(\heartsuit) = \|\Pi_\heartsuit|\psi\rangle\|_2^2$, and leaves the qubit in the state $\Pi_\heartsuit|\psi\rangle / \|\Pi_\heartsuit|\psi\rangle\|_2$. Result $\spadesuit$ happens with probability $p(\spadesuit) = \|\Pi_\spadesuit|\psi\rangle\|_2^2$, and leaves the qubit in the state $\Pi_\spadesuit|\psi\rangle / \|\Pi_\spadesuit|\psi\rangle\|_2$. (In practice the measurement usually destroys the qubit, and often one can ignore the post-measurement state).

a) Show that for any qubit $|\psi\rangle$, and any projectors such that $\Pi_\heartsuit + \Pi_\spadesuit = \mathbb{1}$ we have that $p(\heartsuit) \geq 0, p(\spadesuit) \geq 0$, and $p(\heartsuit) + p(\spadesuit) = 1$. **(0.5 p)**

b) Let $\Pi_\heartsuit = |\theta\rangle\langle\theta|$, for a qubit $|\theta\rangle = \gamma|0\rangle + \delta|1\rangle$. Find $\Pi_\spadesuit$ such that $\{\Pi_\heartsuit, \Pi_\spadesuit\}$ is a measurement. Find a qubit $|\theta^\perp\rangle$ such that $\Pi_\spadesuit = |\theta^\perp\rangle\langle\theta^\perp|$ (the solution is not unique, it is enough to find one). **(0.5 p)**

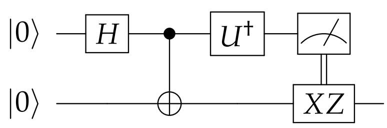
c) In quantum computation measurements are always done in the computational basis $\{|0\rangle, |1\rangle\}$, with results 0, 1, and projectors $\Pi_0 = |0\rangle\langle 0|$ and $\Pi_1 = |1\rangle\langle 1|$. They are represented by the measurement gate . Measurements in other bases are implemented by applying some unitary U to the state before measuring it in the computational basis, as shown in the following circuit:



Find a unitary U such that $p(0) = \|\Pi_0 U|\psi\rangle\|_2^2 = \|\Pi_\heartsuit|\psi\rangle\|_2^2$ and $p(1) = \|\Pi_1 U|\psi\rangle\|_2^2 = \|\Pi_\spadesuit|\psi\rangle\|_2^2$. What are the post-measurement states? **(1 p)**

d) When a measurement is made on only one of a pair of qubits, say the first, the probabilities and post-measurement states are given by $p(\heartsuit) = \|(\Pi_\heartsuit \otimes \mathbb{1})|\psi\rangle|\phi\rangle\|_2^2$ and $(\Pi_\heartsuit \otimes \mathbb{1})|\psi\rangle|\phi\rangle / \|(\Pi_\heartsuit \otimes \mathbb{1})|\psi\rangle|\phi\rangle\|_2$, with the analogous equation for $\spadesuit$. Calculate the post-measurement states for any pair of qubits $|\psi\rangle|\phi\rangle$, for the projectors $\{\Pi_\heartsuit, \Pi_\spadesuit\}$ from item **b)**. **(0.5 p)**

e) Calculate what the following circuit does, with the unitary U from item **c)** (or an arbitrary one, if you couldn't find it), and remembering that the unitary XZ is only applied to the post-measurement state if the result of the measurement was 1.



(1 p)

Hint: It is known as lame teleportation.

2 Pure and mixed states

a) Let ρ be a density matrix (positive semidefinite and trace one). Show that the following statements are equivalent:

(i) $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $p_{i_0} = 1$ for some i_0 or $|\psi_i\rangle = |\psi_0\rangle$ for all i.

(ii) $\rho = |\psi\rangle\langle\psi|$, that is, ρ is a pure state.

(iii) ρ has rank 1

(iv) $\rho^2 = \rho$

(v) $\mathrm{tr}(\rho^2) = 1$

(vi) the spectrum of ρ is given by $\{1, 0, \ldots, 0\}$

(2 p)

Hint: It suffices to show the implications in cyclic order, i.e. (i)⇒(ii)⇒…⇒(vi)⇒(i)

b) Use a criterion of your choice to decide whether the following 1-qubit density matrices, given in the computational basis, are pure or mixed. In the latter case, write them as a convex combination of pure states (the solution is not unique).

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \rho_2 = \frac{1}{2}\begin{pmatrix} 1 & \frac{3}{4} - \frac{i}{4} \\ \frac{3}{4} + \frac{i}{4} & 1 \end{pmatrix},$$

$$\rho_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_4 = \frac{1}{2\sqrt{3}}\begin{pmatrix} \sqrt{3}+1 & 1-i \\ 1+i & \sqrt{3}-1 \end{pmatrix}.$$

(2 p)

3 Separable and entangled states

The reduced density matrix ρ of the first particle of a bipartite state $|\Psi\rangle = \sum_{ij} \alpha_{ij}|ij\rangle$ is given by the partial trace over the second particle, defined as

$$\rho = \mathrm{tr}_2 |\Psi\rangle\langle\Psi| = (\mathbb{1} \otimes \mathrm{tr})\, |\Psi\rangle\langle\Psi| = (\mathbb{1} \otimes \mathrm{tr}) \sum_{ij} \alpha_{ij}\bar{\alpha}_{kl} |i\rangle\langle k| \otimes |j\rangle\langle l| = \sum_{ij} \alpha_{ij}\bar{\alpha}_{kl} |i\rangle\langle k|\, \mathrm{tr}(|j\rangle\langle l|)$$

a) Show that if $|\Psi\rangle = |\psi\rangle|\phi\rangle$ then $\rho = \mathrm{tr}_2 |\Psi\rangle\langle\Psi|$ is a pure state. **(0.5 p)**

b) Therefore if $\rho = \mathrm{tr}_2 |\Psi\rangle\langle\Psi|$ is not pure, then $|\Psi\rangle$ is entangled. Let

$$|\Phi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle.$$

Compute the reduced density matrix of the first particle of $|\Phi\rangle$, and check for which p it is not pure, and therefore $|\Phi\rangle$ is entangled. What is a different way to prepare the same density matrix? **(1 p)**

c) Check whether the following states are entangled or separable:

$$|H\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$$

$$|G\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$|CZ\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$

(1 p)