Quantum Information Theory

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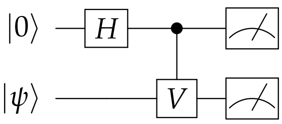
Exercise sheet 3 Due: 2018.05.16 at 12:00

1 POVMs

a) A POVM is a set of positive semidefinite operators $\{E_1, \ldots, E_n\}$ such that $\sum_{i=1}^n E_i = \mathbb{1}$, and it is an useful way to represent the probabilities obtained by measuring a state together with an ancillary system. Consider the following circuit, where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ for **complex** α, β, and $V = |0\rangle\langle +| + |1\rangle\langle -|$:



Calculate the probabilities $p(0,0), p(0,1), p(1,0)$, and $p(1,1)$. Since they are linear functions of $|\psi\rangle\langle\psi|$, by Riesz's theorem one can represent them as a Hilbert-Schmidt inner product

$$p(a,b) = \mathrm{tr}(E_{a,b}|\psi\rangle\langle\psi|)$$

for some operators $E_{a,b}$. Find them, and check that they indeed form a POVM. **(2 points)**

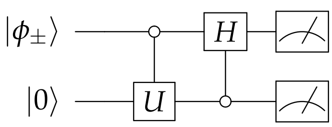
b) The states $|\phi_+\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$ and $|\phi_-\rangle = \cos\theta|0\rangle - \sin\theta|1\rangle$, for $\theta \in [0, \frac{\pi}{4})$, are not orthogonal, and thus no projective measurement can distinguish them perfectly. One can do better, however, with POVMs: it is possible to construct a 3-outcome POVM $\{E_+, E_-, E_?\}$ such that one can be sure that when outcomes $+$ or $-$ are obtained, the state was definitely $|\phi_+\rangle$ or $|\phi_-\rangle$, respectively, at the cost of sometimes obtaining the inconclusive outcome ?. This is known as unambiguous state discrimination. To construct this POVM, let

$$\begin{aligned}
|\phi_+^\perp\rangle &= \sin\theta|0\rangle - \cos\theta|1\rangle \\
|\phi_-^\perp\rangle &= \sin\theta|0\rangle + \cos\theta|1\rangle \\
E_+ &= \lambda|\phi_-^\perp\rangle\langle\phi_-^\perp| \\
E_- &= \lambda|\phi_+^\perp\rangle\langle\phi_+^\perp|
\end{aligned}$$

Find $E_?$ as a function of λ and θ such that this is a valid POVM. Determine the optimal λ as a function of θ such that the probability of an inconclusive result $p(?) = \mathrm{tr}(|\phi_+\rangle\langle\phi_+|E_?) = \mathrm{tr}(|\phi_-\rangle\langle\phi_-|E_?)$ is minimized.

Hint: Remember that all three POVM elements must be positive semidefinite. **(2 points)**

c) Implementing an abstract POVM is always possible (this result is known as Naimark's theorem[1]). Find a unitary U (it is not unique) such that the following circuit produces probabilities equal to the optimal POVM $\{E_+, E_-, E_?\}$ of item **b)**, with $p(0,0) = p(+)$, $p(1,0) = p(-)$, and $p(0,1) = p(?)$. Or if you didn't do it, you can equivalently find a unitary U such $p(0,0||\phi_-\rangle) = p(1,0||\phi_+\rangle) = 0$.



The symbol means that the gate is applied only when the qubit is in the state $|0\rangle$, instead of the usual $|1\rangle$. **(3 points)**

[1] See the book *Geometry of Quantum States*, by Życzkowski & Bengtsson, for a proof.

d) **(Bonus item)** Use Riesz's theorem to derive a formula that gives the POVM elements corresponding to a circuit, and use it to calculate the POVM elements from item **c)**. **(2 points)**

2 Statistics of Bell experiments

For $i = 1, \ldots, n$, let $(M_1^{(i)}, M_2^{(i)}, N_1^{(i)}, N_2^{(i)})$ be a vector of four numbers in $\{\pm 1\}$. (We interpret it as the – partly unmeasured – outcomes of the four possible measurements performed in the ith run of a Bell-type experiment). The goal of this exercise is to get a feeling how well the quantity

$$C = M_1 N_1 + M_1 N_2 + M_2 N_1 - M_2 N_2$$

can be estimated if we have access only to a randomly chosen pair $(M_{X^{(i)}}^{(i)}, N_{Y^{(i)}}^{(i)})$ of measurements per run. Here, we assume that X and Y are independent random variables that take on the values $\{1, 2\}$ with probability $\frac{1}{2}$.

We'll make use of the *Chernoff-Hoeffding inequality* (a proof of which you are encouraged to look up). It says that if $A_1, \ldots, A_n$ are independent random variables that take values in $[-4, 4]$, and

$$S_n := \frac{1}{n} \sum_{i=1}^{n} A_i$$

is their mean, then, for all $\varepsilon > 0$,

$$\Pr\Big[\big| S_n - \langle S_n \rangle \big| \geq \varepsilon \Big] \leq 2e^{-\frac{n\varepsilon^2}{32}}.$$

Here, $\langle S_n \rangle$ is the expectation value of S_n. In other words, the bound says that the probability that S_n will deviate from its expected value is exponentially small in the squared deviation.

a) Focus on the first summand in C. Set

$$A_i = \begin{cases} 4M_1^{(i)} N_1^{(i)} & \text{if } X^{(i)} = Y^{(i)} = 1 \\ 0 & \text{else} \end{cases}$$

Show that $\langle S_n \rangle = \frac{1}{n} \sum_{i=1}^{n} M_1^{(i)} N_1^{(i)}$. This justifies the use of S_n as an estimate for the mean value of $M_1^{(i)} N_1^{(i)}$ over the table. The other summands can be estimated similarly. **(1 points)**

b) Now assume that the estimation procedure above has been performed for $n = 10\,000$ runs and resulted in an estimate of 2.8 for the mean of C over the table. Of course, it could be that the true mean value of C is actually smaller than or equal to 2 and that the apparent larger value is a statistical fluke. Prove that this is less likely than winning the "6 out of 49" lottery.

Hint: Use that at all four estimated summands must deviate by at least 0.2 from their actual mean. **(2 points)**