Quantum Information Theory

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Exercise sheet 4 **Due: 2018.06.06 at 12:00**

(sheets delivered before 2018.06.05 at 12:00 will be corrected before the exercise class)

1 Completely Positive and Trace Preserving (CPTP) maps

a) Let ρ^{AB} be a bipartite operator with dimensions d_A and d_B. Show that if $\rho^{AB} \geq 0$, that is, if ρ^{AB} is positive semidefinite, then its partial trace $\operatorname{tr}_2 \rho^{AB}$ is also positive semidefinite, that is, $\operatorname{tr}_2 \rho^{AB} \geq 0$.

Hint: Use the fact that an operator A is positive semidefinite iff $\langle v|A|v\rangle \geq 0 \quad \forall |v\rangle$. **(1 points)**

b) The completely depolarising map acting on a quantum state ρ of dimension d is defined as

$$\Lambda(\rho) = p\rho + (1-p)\frac{\mathbb{1}}{d}\operatorname{tr}(\rho),$$

for $p \in [0,1]$. It is a simple and useful way to model noise: you get the state ρ with probability p, or the maximally mixed state $\frac{\mathbb{1}}{d}$ with probability $1-p$. Show that this map is indeed completely positive and trace preserving via two methods: first directly, using the result of item **a)**, and then by checking that its Choi-Jamiołkowski representation is positive semidefinite.

Reminder: The Choi-Jamiołkowski representation of a map Λ acting on a state of dimension d is given by $\mathfrak{C}(\Lambda) = (\mathbb{1}_d \otimes \Lambda)(\sum_{i,j=0}^{d-1} |ii\rangle\langle jj|) = \sum_{i,j=0}^{d-1} |i\rangle\langle j| \otimes \Lambda(|i\rangle\langle j|)$. A map Λ is completely positive iff $(\mathbb{1}_{d_A} \otimes \Lambda)(\rho^{AB}) \geq 0$ for all $\rho^{AB} \geq 0$ of arbitrary dimension d_A. **(2.5 points)**

c) As all CPTP maps, the completely depolarising map can be written as

$$\Lambda(\rho) = \sum_i K_i \rho K_i^\dagger$$

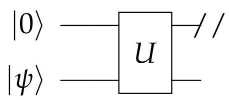
for some operators K_i such that $\sum_i K_i^\dagger K_i = \mathbb{1}$. This is known as the operator-sum representation or Kraus decomposition. Find a Kraus decomposition (it is not unique) for the completely depolarising map with $d = 2$ by diagonalising its Choi-Jamiołkowski representation.

Reminder: If the Choi-Jamiołkowski representation has eigenvalue λ_i with eigenvector $|\lambda_i\rangle$, then the Kraus operators are obtaining by "reshaping" them: $\sqrt{\lambda_i}|\lambda_i\rangle = \sum_{kl} \alpha_{kl}^{(i)} |k\rangle|l\rangle$ becomes $K_i = \sum_{kl} \alpha_{kl}^{(i)} |k\rangle\langle l|$. **(2 points)**

d) **(Bonus item)** Find a Kraus decomposition for the completely depolarising map with arbitrary d by whichever method you can. **(1 points)**

2 Stinespring dilation

Every CPTP map can actually be implemented, by applying an unitary U to the system of interest $|\psi\rangle$ together with an ancilla in a fixed state $|0\rangle$ and then discarding the ancilla, as shown in the following circuit:



The symbol // means taking the partial trace, and the wires here do not represent qubits as usual, but qudits of arbitrary dimension.

This result is known as the Stinespring dilation theorem.

a) Find a Kraus decomposition (it is not unique) for the CPTP map acting on $|\psi\rangle\langle\psi|$ implemented by this circuit.

Hint: The strange-looking identiy $(|u\rangle \otimes \mathbb{1})A(\langle v| \otimes \mathbb{1}) = |u\rangle\langle v| \otimes A$ is useful. **(1 points)**

b) Check that if U is such that $U|0\rangle|\psi\rangle = \sum_i |i\rangle K_i|\psi\rangle$, then this circuit implements the CPTP map with Kraus uperators K_i. Check furthermore that $\langle 0|\langle\phi|U^\dagger U|0\rangle|\psi\rangle = \langle\phi|\psi\rangle$. Why does such a U always exist? What is its minimal dimension? **(1 points)**

c) Find a unitary U (it is not unique) such that this circuit implements the completely depolarising map for $d = 2$. Attention! The dimension of the ancilla is not equal to 2, only the dimension of $|\psi\rangle$. **(2.5 points)**

3 Peres-Horodecki criterion (bonus exercise)

If one applies the completely depolarising map to the singlet state $|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, the result is the Werner state

$$W(p) = p|\psi_-\rangle\langle\psi_-| + (1-p)\frac{\mathbb{1}}{4},$$

named after Reinhard Werner. An important question in quantum information is to determine how much noise the quantum state can withstand before becoming useless for information processing tasks. In this case, we want to know for which range of p the state $W(p)$ is entangled. For that, we first need to define what it means for a mixed state ρ^{AB} to be separable: it is iff it can be written as a convex combination of product states, that is,

$$\rho^{AB} = \sum_i \lambda_i \rho_i^A \otimes \rho_i^B,$$

for some quantum states ρ_i^A and ρ_i^B, and some $\lambda_i \geq 0$ such that $\sum_i \lambda_i = 1$. A mixed state is entangled iff it is not separable.

a) Check that if a bipartite state ρ is separable, then its partial tranpose $\rho^\Gamma = (\mathbb{1} \otimes T)(\rho)$ is positive semidefinite. **(0.5 points)**

b) Calculate for which range of p the matrix $W(p)^\Gamma$ is not positive semidefinite, and therefore the state $W(p)$ is entangled. **(1 points)**