Quantum Information Theory

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Exercise sheet 5 **Due: 2018.06.20 at 12:00**

(sheets delivered before 2018.06.19 at 12:00 will be corrected before the exercise class)

1 Schmidt decomposition

Every pure bipartite quantum state $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle,$$

for $\lambda_i \geq 0$ and orthonormal bases $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$. This is known as the Schmidt decomposition, and λ_i are the Schmidt coefficients.

a) Calculate the Schmidt coefficients of the states

$$|J\rangle = \frac{3}{5}|00\rangle + \frac{4}{5}|10\rangle$$
$$|H\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$$
$$|F\rangle = \frac{1}{\sqrt{7}}(|00\rangle + 2|01\rangle + |10\rangle + |11\rangle)$$

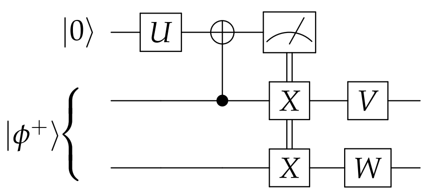
Reminder: You can find how to do the Schmidt decomposition in page 109 of "Quantum Computation and Quantum Information" by Nielsen & Chuang. **(1 points)**

b) Show that if the bipartite states (of arbitrary dimension) $|\psi\rangle$ and $|\phi\rangle$ have the same Schmidt coefficients, then there exists unitaries V and W such that

$$|\psi\rangle = V \otimes W|\phi\rangle$$

(1 points)

c) Let $|\psi\rangle$ be an arbitrary two-qubit state. Find U, V, and W such that $|\psi\rangle$ is produced in the two lower wires of the following circuit:



Here $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is the maximally entangled state and X is the Pauli matrix. X is applied to the two lower qubits only if the result of the measurement is 1. The first qubit can be discarded after the measurement. If we say that the first and second qubits belong to Alice, and the third to Bob, than Alice and Bob are only doing local operations on $|\phi^+\rangle$ and exchanging classical communication. This shows how powerful a resource $|\phi^+\rangle$ is.

(2 points)

d) Conversely, show that if instead of $|\phi^+\rangle$ we input a non-maximally entangled state (e.g. $|P\rangle = \frac{3}{5}|00\rangle + \frac{4}{5}|11\rangle$), it is not possible to prepare an arbitrary two-qubit state in the two lower wires, by finding an specific state which cannot be done. **(2.25 points)**

2 Tripartite entanglement

One might hope that a Schmidt decomposition also exists for tripartite states, i.e., that any state could be written in the form

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle|i_C\rangle$$

for $\lambda_i \geq 0$ and orthonormal bases $\{|i_A\rangle\}, \{|i_B\rangle\}$, and $\{|i_C\rangle\}$.

a) Assuming that $|\psi\rangle$ is of this form, calculate the spectrum of the reduced density matrices $\rho^A = \mathrm{tr}_{23} |\psi\rangle\langle\psi|$, $\rho^B = \mathrm{tr}_{13} |\psi\rangle\langle\psi|$, and $\rho^C = \mathrm{tr}_{12} |\psi\rangle\langle\psi|$. **(0.5 points)**

b) Using the result of item **a)**, or whatever method you prefer, prove that

$$|\phi\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|011\rangle + \frac{1}{\sqrt{2}}|100\rangle$$

cannot be written in the Schmidt form. **(1.25 points)**

c) Consider the quantum states

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$
$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

Calculate the reduced density matrices $\mathrm{tr}_3 |W\rangle\langle W|$ and $\mathrm{tr}_3 |GHZ\rangle\langle GHZ|$, and test whether they are entangled using the Peres-Horodecki criterion introduced in the previous exercise sheet (in general one can only guarantee that if the partial transpose is not positive semidefinite then the state is entangled. But for two-qubit states like the ones here the converse holds: if the partial transpose is positive semidefinite then the state is separable). What does this tell you about the resistance of the entanglement of these states against particle loss? **(2 points)**