Quantum Information Theory

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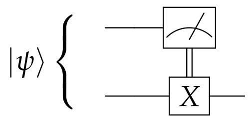
Exercise sheet 6 **Due: 2018.07.04 at 12:00**

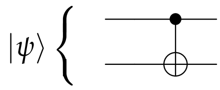
(sheets delivered before 2018.07.03 at 12:00 will be corrected before the exercise class)

1 Review

a) Calculate the effect of the circuits

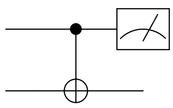


and



on an arbitrary two-qubit state $|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$, for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. For clarity, do not discard the measured qubit. **(0.5 points)**

b) Show that the circuits



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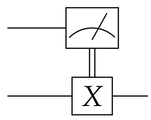


are equivalent. This equality explains why one often gets the correct result if one applies a CNOT instead of making a measurement and applying a classically controlled X gate. **(0.5 points)**

2 A stupid code

a) One might consider using two qubits instead of three to correct against flips of single qubits. The codewords would be

$$|0\rangle \mapsto |0_L\rangle = |00\rangle$$
$$|1\rangle \mapsto |1_L\rangle = |11\rangle$$

so that a flip of a single qubit does map it out of the codespace, and measuring the syndrome Z_1Z_2 tells us if an error occurred (result -1), or we are still in the codespace (result $+1$). Without doing any calculations, explain why this doesn't actually work. **(0.5 points)**

Reminder: The notation A_iB_j is used a lot in error correction, and means that operator A acts on qubit i, operator B acts on qubit j, and identity acts on the rest. In this case $Z_1Z_2 = Z \otimes Z$.

b) A more sophisticated attempt at a two-qubit code is the stabilizer code with stabilizer group $S = \langle X_1X_2, Z_1Z_2\rangle$. Check that this is in fact a stabilizer code by checking that the generators X_1X_2 and Z_1Z_2 are independent and commute, and that $-\mathbb{1} \notin S$ by computing the whole group S. **(0.5 points)**

c) Characterize the three kinds of error for the stabilizer code S from item **b)**. **(1 points)**

Reminder: The three kinds of error $E \in G_n$ (where G_n is the Pauli group) that can occur are 1) E commutes with all elements of S and in fact belongs to S, so that it is not an error at all. 2) E anti-commutes with at least one element of S, so it can be corrected. 3) E commutes with all elements of S and does not belong to S, so it cannot be corrected.

d) Find out what the codespace is via computing the projector onto it $P_S = \frac{1}{|S|}\sum_{s\in S} s$. What is going on? For a sanity check, remember that the dimension of the codespace is given by 2^{n-k}, where n is the number of qubits and k the number of generators. **(1 points)**

3 The toric code

The toric code is a stabilizer code defined on a two-dimensional lattice that has the topology of a torus: the right edge (dashed) is identified with the left edge, and the lower edge (dashed) is identified with the higher edge. The physical qubits are the disks on the lattice, and qubits with the same number are again identified. For concreteness we are dealing here with a 3×3 lattice, but the toric code can be defined for a lattice as large as you want. The generators of its stabilizer group T are the star operators $A_V = \prod_{v\in V} X_v$, which are the product of Pauli X operators acting on the qubits around a vertex V, and the plaquette operators $B_S = \prod_{s\in S} Z_V$, which are the product of Pauli Z operators acting on the qubits on the perimeter of a square S.

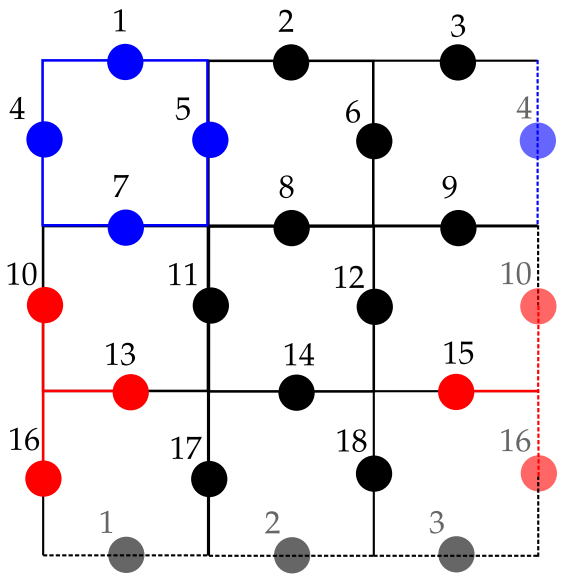
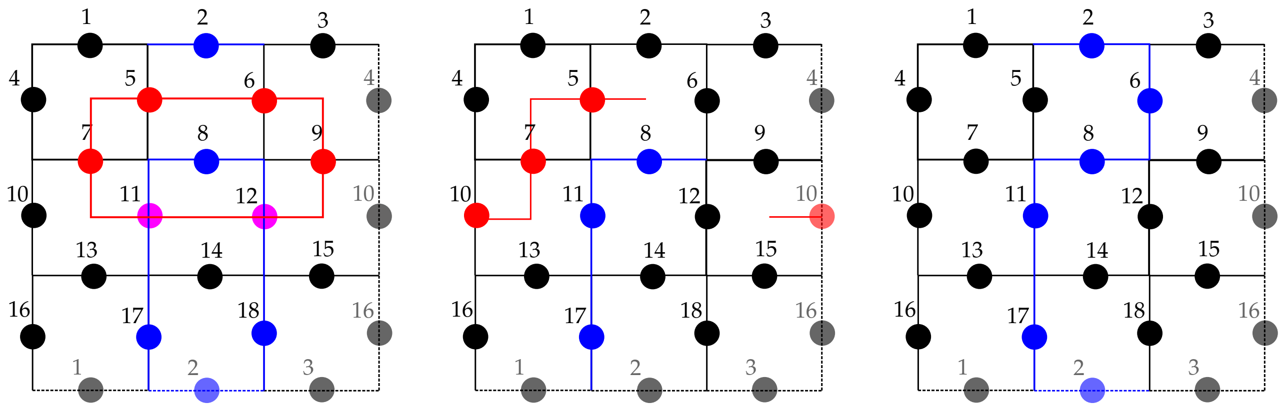


Figure 1: A 18-qubit lattice implementing the toric code. The blue square represents the plaquette operator $Z_1Z_4Z_5Z_7$ and the red edges represent the star operator $X_{10}X_{15}X_{13}X_{16}$.

a) Check that the generators in fact commute and that $-\mathbb{1}$ is not in the stabilizer group T, so that this is a valid stabilizer code.

Hint: You do not need to compute the whole stabilizer group T. **(0.5 points)**

b) The generators are not independent. To see that, compute the product of all 9 star operators, and the product of all 9 plaquette operators. Using this result, show explicitly two different ways of representing the plaquette operator $Z_1Z_4Z_5Z_7$ as a product of plaquette operators. Since these are the only constraints that appear, now you can calculate the number of independent generators. What is the dimension of the codespace? **(1.5 points)**



(a) Closed loops (b) Open lines (c) Irreducible loop

Figure 2: Errors in the toric code

c) Consider the errors shown in figure 2(a), where red qubits have been hit with a X, blue qubits have been hit with a Z, and purple qubits have been hit with both an X and a Z. Show that these are errors of the first kind, by explicitly showing which product of plaquettes and stars generates these "errors". The red and blue lines are there as a hint. (In fact, all closed loops on the toric code are errors of the first kind.) **(1.5 points)**

d) Consider now the errors shown in figure 2(b). Show that these are errors of the second kind, correctable, by finding the outcome of the syndrome measurements that identity these errors, and finding which operation must be applied to the physical qubits in order to map these errors into an error of the first kind.

Hint: First look for all plaquettes and stars that anticommute with these errors, show that there exists another error of the second kind that also anticommutes with the same plaquettes and stars (there are several, but finding one is enough), and find the operation that will map either of these errors into an error of the first kind. (In fact, all open lines on the toric code are errors of the second kind.)

(1.5 points)

e) Consider now the errors shown in figure 2(c). Show that these are errors of the third kind, by showing that they commute with all plaquettes and stars, but are not in the stabilizer group T.

Hint: Argue using the key topological property of the torus, that loops that wrap around it like this cannot be deformed into closed loops. Such loops might be easier to visualize in a three-dimensional representation of the torus, as in figure 3. **(1 points)**

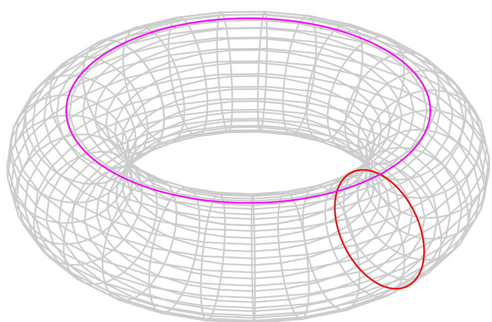


Figure 3: Two irreducible loops on a torus.