Quantum Mechanics

David Gross, Mateus Araújo

Sheet 1 Due: 09.04 um 12 Uhr

Hints for doing exercises

- Homepage: http://www.thp.uni-koeln.de/gross/qm-summer19.html
- The exercises will always be published Tuesdays at 12:00 on the website above. The solutions should be delivered one week later in the mailboxes of the old building..
- Arrange all pages in the correct order and write down your name, the number of your exercise group, and the name of your tutor in the first page.
- Please deliver the solutions in groups of three. We strongly encourage you to work out the solutions in this configuration and discuss possible ways to find a solution.
- If you have difficulties with the subject matter, try to promptly address them. Your questions can be answered: in the literature, by your colleagues, the question time Mondays after the lecture, the exercise classes, or the teachers.

1 Linear Algebra with Bra-Ket notation

These are example exercises for the tutors. They are not worth any points, but everyone should be able to solve them. In what follows let $\{|\psi_i\rangle\}_{i=1}^d$ be an orthonormal basis.

a) Let $|\alpha\rangle = \sum_i c_i |\psi_i\rangle$ be a vector. Show that: $c_{i_0} = \langle\psi_{i_0}|\alpha\rangle$ and $\|\alpha\|_2^2 = \sum_i |c_i|^2$, where $\|\cdot\|_2$ is the Euclidean norm.

b) Show the *completeness relation* $\sum_i |\psi_i\rangle\langle\psi_i| = \mathbb{1}$, where $\mathbb{1}$ is the *identity operator*, that maps every vector to itself.

c) The *trace* from an operator A can be defined as: $\operatorname{tr} A = \sum_i \langle\psi_i|A|\psi_i\rangle$. Show that, for any two operators A and B we have that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. (Hint: put the completeness relation in). Conclude that: $\operatorname{tr}(ABC) = \operatorname{tr}(BCA)$ (*cyclical invariance of the trace*) and $\operatorname{tr}(ABA^{-1}) = \operatorname{tr} B$ for invertible A (*invariance under conjugation*). Conclude also that $\operatorname{tr} A$ is equal to the sum of the eigenvalues of A counted with their multiplicity. (In particular the trace does not depend on the basis which we used to define it).

d) Let A be an operator. The adjoint operator $A^\dagger$ is defined through the relation $\langle\alpha|A^\dagger|\beta\rangle = \overline{\langle\beta|A|\alpha\rangle}$. Why does this relation define $A^\dagger$ completely? Let $|\psi\rangle$ be an eigenvector of A with eigenvalue λ: $A|\alpha\rangle = \lambda|\alpha\rangle$. Show that $\langle\alpha|A^\dagger = \bar{\lambda}\langle\alpha|$. Show that if A is self-adjoint, that is if $A = A^\dagger$, then all eigenvalues of A are real.

2 Uncertainty relations (5 P)

In this exercise we shall prove uncertainty relations that show that the x and z components of the spin of a particle cannot be simultaneously well-defined. The spin observables in the x and z

directions in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis are given by the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) (1 P) Let A be an observable. The *variance* of A with respect to the state $|\psi\rangle$ is given by

$$\mathrm{Var}_\psi(A) = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2.$$

Show that $\mathrm{Var}_\psi(A) = 0$ when $|\psi\rangle$ is a normalised eigenvector of A.

b) (2 P) Let $|\psi_\alpha\rangle = \cos(\alpha/2)|\uparrow\rangle + \sin(\alpha/2)|\downarrow\rangle$. Compute $\mathrm{Var}_{\psi_\alpha}(\sigma_z)$ (please simplify with trigonometrical formulas). For which values of $\alpha \in [0, 2\pi)$ is $|\psi_\alpha\rangle$ an eigenvector of σ_x? What is the variance of σ_z for these vectors?

c) (2 P) It should now seem plausible that the variance is a measure of how "indefinite" some physical quantity is. Show the *uncertainty relation*

$$\mathrm{Var}_{\psi_\alpha}(\sigma_x) + \mathrm{Var}_{\psi_\alpha}(\sigma_z) = 1.$$

Describe in no more than two sentences what you can conclude from this formula.

3 (Bonus exercise) More uncertainty relations (2 P)

Show that for an arbitrary spin-$\frac{1}{2}$ state $|\phi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$, the following uncertainty relation holds:

$$\mathrm{Var}_\phi(\sigma_x) + \mathrm{Var}_\phi(\sigma_y) + \mathrm{Var}_\phi(\sigma_z) = 2.$$

Here σ_y is the spin observable in the y direction given by

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$