Quantenmechanik

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Sheet 10 **Due: 18.06 um 12 Uhr**

1 Quantum correlations (6 P)

In the lecture we learned about the famous singlet state, that describes two Spin-1/2 particles with total angular momentum zero. We are going to analyse a couple of its properties.

a) **(0,5 P)** The observable $S_\alpha = \sin(\alpha)\sigma_x + \cos(\alpha)\sigma_z$ measures the angular momentum of a Spin-$\frac{1}{2}$ particle in a direction in the $x - z$ plane, in an angle α relative to the z axis. To see that, show that the expectation value of S_α calculated with the quantum state $|\psi(\theta)\rangle = \cos(\theta/2)|\uparrow\rangle + \sin(\theta/2)|\downarrow\rangle$, is given by

$$\langle\psi(\theta)|S_\alpha|\psi(\theta)\rangle = \cos(\theta - \alpha).$$

b) **(1 P)** The singlet state can be written as

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle).$$

We can calculate the expectation value of an angular momentum measurement on the first (respectively second) particle by computing the expectation values of the observables S_α^1 (respectively S_α^2). The action of S_α^1 on the singlet is given by

$$S_\alpha^1|\Psi^-\rangle = \frac{1}{\sqrt{2}}\Big((S_\alpha|\uparrow\rangle)|\downarrow\rangle - (S_\alpha|\downarrow\rangle)|\uparrow\rangle\Big),$$

and analogously the action of S_α^2 by

$$S_\alpha^2|\Psi^-\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle(S_\alpha|\downarrow\rangle) - |\downarrow\rangle(S_\alpha|\uparrow\rangle)\Big).$$

Show that

$$\langle\Psi^-|S_\alpha^1|\Psi^-\rangle = 0 = \langle\Psi^-|S_\alpha^2|\Psi^-\rangle.$$

What does this equation means physically?

c) **(2,5 P)** Imagine that two experimentalists, Alice and Bob, want to measure the *correlations* between the particles. For that they measure S_α^1 and S_β^2 simultaneously. Show that

$$\langle\Psi^-|S_\alpha^1 S_\beta^2|\Psi^-\rangle = -\cos(\alpha - \beta),$$

and explain the physical reason why the expectation value for $\alpha = \beta$ must be equal to -1.

d) **(2 P)** The singlet is an *entangled* quantum state, that is, it is not a product from a quantum state $|\psi(\theta_1)\rangle$ for the first particle with a quantum state $|\psi(\theta_2)\rangle$ for the second particle. To prove that, show that

$$\langle\psi(\theta_1)|\langle\psi(\theta_2)|S_\alpha^1 S_\beta^2|\psi(\theta_1)\rangle|\psi(\theta_2)\rangle = \cos(\theta_1 - \alpha)\cos(\theta_2 - \beta),$$

and that no angles θ_1, θ_2 exist so that

$$\cos(\theta_1 - \alpha)\cos(\theta_2 - \beta) = -\cos(\alpha - \beta).$$

for all values of α, β.

Hint: This equality must be valid for a fixed pair of θ_1, θ_2, that does not depend on α and β. Find θ_1, θ_2 for a particular value of α and β and show that these do not work for some other value of α and β.

2 Classical analogy (4 P)

The correlation that you have calculated in exercise **1c** is a bit peculiar. To see what's strange about it, here we shall develop a classical analog of the singlet state, and see that its correlations are qualitatively different. Our model is a bomb with initial angular momentum zero, that explodes in two pieces, one with angular momentum $J_1 = (\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi)$ and another with angular momentum $J_2 = -J_1$. Alice and Bob perform again measurements on J_1 and J_2.

a) **(1 P)** Here we shall calculate the local expectation values, in analogy with exercise **1b**. Alice measures the angular momentum of her half with respect to a direction $m_\alpha = (\cos\alpha, \sin\alpha)$ for $\alpha \in [0, 2\pi)$. The result is

$$r_\alpha = \operatorname{sgn}(m_\alpha \cdot J_1),$$

where $\operatorname{sgn}(x)$ is the sign function. The expectation value of r_α for uniformly distributed angular momenta is given by

$$\langle r_\alpha \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \operatorname{sgn}(m_\alpha \cdot J_1).$$

Show that $\langle r_\alpha \rangle = 0$, that is, the local expectation values are the same as in the quantum case.

b) **(2 P)** In analogy with exercise **1c**, here we shall calculated the *correlated* expectation value of Alice and Bob. It is given by

$$\langle r_\alpha r_\beta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \operatorname{sgn}(m_\alpha \cdot J_1) \operatorname{sgn}(m_\beta \cdot J_2),$$

again for uniformly distributed angular momenta. Show that

$$\langle r_\alpha r_\beta \rangle = -1 + \frac{2\phi_{\alpha\beta}}{\pi},$$

where $\phi_{\alpha\beta} = \min\{|\alpha - \beta|, 2\pi - |\alpha - \beta|\}$ is the angle between m_α and m_β.

Hint: Keep in mind that $J_2 = (\cos(\theta + \pi), \sin(\theta + \pi))$.

c) **(1 P)** Sketch the graphs of $-1 + \frac{2\phi_{\alpha\beta}}{\pi}$ and $-\cos(\alpha - \beta)$ to show that the quantum and classical correlations are *not* the same.