Quantenmechanik

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Sheet 13 **Due: 09.07 um 12 Uhr**

Remark: All points in this sheet are bonus points. Not because the exercises are particularly difficult, but because the content here is not necessary for the exam. We find this subject rather interesting, though, and hope that you will also.

1 Decoherence (10 P)

Why do we see a classical world? One answer lies on *decoherence* – the ubiquitous and (almost) unavoidable interaction with the environment, that washes out superpositions on macroscopic scales. We'll now see how.

As a simple, but realistic model for it we will consider a macroscopic dust mote with position degree of freedom $|x\rangle$ that scatters a photon in a coherent state $|L\rangle$. The dust mote stays unchanged, whereas the photon experiences a displacement of $D(x\sqrt{2\Lambda})$, where $D(\ell) = e^{\ell(a^\dagger - a)}$ is the displacement operator from exercise **1e** from sheet 7. The scattering in every position x is then given by

$$|x\rangle|L\rangle \mapsto |x\rangle D(x\sqrt{2\Lambda})|L\rangle.$$

a) **(1 P)** Show that

$$D(\ell_1)D(\ell_2) = D(\ell_1 + \ell_2)$$

b) **(1 P)** Assume that the dust mote is initially in a superposition of two places x_1 and x_2, therefore with a quantum state

$$|\psi\rangle = \alpha|x_1\rangle + \beta|x_2\rangle.$$

Show that after the scattering the dust mote-photon combined state is given by

$$|\Psi\rangle = \alpha|x_1\rangle|L + x_1\sqrt{2\Lambda}\rangle + \beta|x_2\rangle|L + x_2\sqrt{2\Lambda}\rangle.$$

Hint: $|L\rangle = D(L)|0\rangle$, for coherent states $|L\rangle$ and $|0\rangle$.

c) **(2 P)** The *partial trace* over the second part of an arbitrary bipartite quantum state $|\gamma\rangle = \sum_{ij} \alpha_{ij}|\phi_i\rangle|\varphi_j\rangle$ is defined as

$$\operatorname{tr}_2(|\gamma\rangle\langle\gamma|) = \operatorname{tr}_2\left(\sum_{ijkl} \alpha_{ij}\bar{\alpha}_{kl}|\phi_i\rangle\langle\phi_k| \otimes |\varphi_l\rangle\langle\varphi_j|\right) = \sum_{ijkl} \alpha_{ij}\bar{\alpha}_{kl}|\phi_i\rangle\langle\phi_k|\langle\varphi_l|\varphi_j\rangle.$$

To compute the density matrix of the dust mote, we need to perform the partial trace over the photon. Show that the density matrix in the $\{|x_1\rangle, |x_2\rangle\}$ is given by

$$\rho = \operatorname{tr}_2 |\Psi\rangle\langle\Psi| = \begin{pmatrix} |\alpha|^2 & \alpha\bar{\beta}e^{-\Lambda(x_1-x_2)^2} \\ \bar{\alpha}\beta e^{-\Lambda(x_1-x_2)^2} & |\beta|^2 \end{pmatrix}.$$

Hint: The inner product between two coherent states $|a\rangle$ and $|b\rangle$ is $\langle a|b\rangle = e^{-\frac{1}{2}(|a|^2+|b|^2-2\bar{a}b)}$. Here Λ, x_1, x_2, L are real numbers.

d) **(2 P)** The parameter Λ is known as "localisation rate", and it describes how strong the interaction between dust mote and photon is. What happens with the off-diagonal elements of ρ when the interaction is weak, that is, when $\Lambda \approx 0$? Can one observe interference in this case? And what happens when the interference is strong, that is, $\Lambda \gg 1/(x_1 - x_2)^2$? What happens in these two cases with the entanglement of $|\Psi\rangle$? For simplicity, assume that $\alpha = \beta = 1/\sqrt{2}$.

e) **(2 P)** Using scattering theory one can calculate Λ. For a black body radiation with temperature T (in Kelvin) that interacts with a dust mote with diameter a (in centimetres), the localisation rate (per second) is given by

$$\Lambda \approx 5 \times 10^{19} T^9 a^6.$$

Using realistic values for a and T in space and room temperature, compute the largest distance $|x_1 - x_2|$ in which interference is still visible.

f) **(2 P)** To generalise the model, we assume that the dust mote is initially described by an arbitrary wave function $\varphi(x)$ (instead of just a superposition of two places). The scattering in this case is described by the transformation

$$\int \mathrm{d}x \varphi(x) |x\rangle |L\rangle \mapsto \int \mathrm{d}x \varphi(x) |x\rangle D(x\sqrt{2\Lambda}) |L\rangle$$

and the density matrix of the dust mote is analogously defined:

$$\begin{aligned} \rho &= \mathrm{tr}_2 \int \mathrm{d}x\, \mathrm{d}x' \varphi(x) \bar{\varphi}(x') |x\rangle\langle x'| \otimes |L + x\sqrt{2\Lambda}\rangle\langle L + x'\sqrt{2\Lambda}| \\ &= \int \mathrm{d}x\, \mathrm{d}x' \varphi(x) \bar{\varphi}(x') |x\rangle\langle x'| \left\langle L + x'\sqrt{2\Lambda} \middle| L + x\sqrt{2\Lambda} \right\rangle. \end{aligned}$$

Show that

$$\langle x|\rho|x'\rangle = \varphi(x)\bar{\varphi}(x') e^{-\Lambda(x-x')^2}.$$

Assume that $\varphi(x)$ is the Schrödinger cat state for $t = 0$ from sheet 7. Plot it as function of x, x' in 3D with a computer, to illustrate the effect of decoherence.