Quantenmechanik

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Sheet 2 **Due: 16.04 um 12 Uhr**

1 Quantum state space (5 P)

In this exercise we shall develop an well-known representation for the state space of spin-$\frac{1}{2}$ particles. It is known as Bloch sphere (named after Felix Bloch). It is one of the very few cases where one can completely visualise a quantum state, and with that illustrate many fundamental phenomena.

a) **(1 P)** The quantum state of a Spin-$\frac{1}{2}$ particle is a vector $|\psi\rangle \in \mathbb{C}^2$ that fulfills the normalisation condition $\||\psi\rangle\|_2 = 1$. Show that any such vector can be written as

$$|\psi\rangle = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) \\ e^{i\beta}\sin(\theta/2) \end{pmatrix},$$

for $\theta \in [0, \pi]$ and $\alpha, \beta \in [0, 2\pi)$.

b) **(1 P)** Show that the quantum states $|\psi\rangle$ and $e^{i\gamma}|\psi\rangle$ give the same probabilities for any possible measurement. This means that $|\psi\rangle$ and $e^{i\gamma}|\psi\rangle$ are physically equivalent.

c) **(2 P)** This equivalence allows us to choose γ so that the representation of $|\psi\rangle$ becomes simpler. Let $\gamma = -\alpha$ and $\varphi = \beta - \alpha$. With that

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\varphi}\sin(\theta/2) \end{pmatrix},$$

and two different θ, φ represent two different quantum states which are *not* physically equivalent.

Using this representation, compute the expectation values $\langle\psi|\sigma_x|\psi\rangle$, $\langle\psi|\sigma_y|\psi\rangle$, and $\langle\psi|\sigma_z|\psi\rangle$, where σ_x, σ_y, and σ_z are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{und} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and show that the vectors

$$\vec{v} = \begin{pmatrix} \langle\psi|\sigma_x|\psi\rangle \\ \langle\psi|\sigma_y|\psi\rangle \\ \langle\psi|\sigma_z|\psi\rangle \end{pmatrix}$$

form a sphere with centre $\vec{0}$ and radius 1. This is the Bloch sphere.

Reminder: A sphere with centre $\vec{v}_0$ and radius r ist the set of points $\vec{v}$ for which $\|\vec{v} - \vec{v}_0\|_2 = r$.

d) **(1 P)** Where in the Bloch sphere are the eigenstates of σ_x, σ_y, and σ_z represented? **Reminder:** All three matrices have eigenvalues ± 1, and the eigenstates are $(|0\rangle \pm |1\rangle)/\sqrt{2}$, $(|0\rangle \pm i|1\rangle)/\sqrt{2}$, and $|0\rangle$ and $|1\rangle$.

2 Movement around the Bloch sphere (5 P)

In the exercise we shall compute and visualize the dynamics of a Spin-$\frac{1}{2}$ quantum state. These dynamics describe, for example, how a Spin-$\frac{1}{2}$ particle behaves in a magnetic field. The representation worked out here is used in the theory of quantum computation and NMR spectroscopy

a) **(1 P)** Show that

$$\exp(it\sigma_x) = \cos(t)\mathbb{1} + i\sin(t)\sigma_x \quad \text{and} \quad \exp(it\sigma_z) = \cos(t)\mathbb{1} + i\sin(t)\sigma_z,$$

where $t \in \mathbb{R}$.

Hint: The exponential of a matrix A is defined as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

b) **(2 P)** Let $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$ be an arbitrary Spin-$\frac{1}{2}$ state, where $|0\rangle$ and $|1\rangle$ are the eigenstates of σ_z. Compute the Bloch sphere representation of

$$\exp(it\sigma_z)|\psi\rangle$$

according to exercise **1c**. Plot the result (with e.g. Julia or Wolfram Alpha) as a function of t for some specific $|\psi\rangle \neq |0\rangle, |1\rangle$, and describe it with words (you don't need to print the graph!)

c) **(2 P)** Let $|\psi\rangle = \cos(\theta/2)|+\rangle + e^{i\varphi}\sin(\theta/2)|-\rangle$ be an arbitrary Spin-$\frac{1}{2}$ state, where $|\pm\rangle$ are the eigenstates of σ_x. Compute the Bloch sphere representation of

$$\exp(it\sigma_x)|\psi\rangle$$

according to exercise **1c**. Plot the result (with e.g. Julia or Wolfram Alpha) as a function of t for some specific $|\psi\rangle \neq |\pm\rangle$, and describe it with words (you don't need to print the graph!)

3 (Bonus exercise) Rotated Bloch sphere (2 P)

Let $|\psi\rangle = \cos(\theta/2)|+u\rangle + e^{i\varphi}\sin(\theta/2)|-u\rangle$ be an arbitrary Spin-$\frac{1}{2}$ quantum state, where $|\pm u\rangle$ are the eigenstates of $a\sigma_x + b\sigma_y + c\sigma_z$, $a, b, c \in \mathbb{R}$ and $a^2 + b^2 + c^2 = 1$. Compute the *rotated* Bloch sphere representation of

$$\exp\left[it\left(a\sigma_x + b\sigma_y + c\sigma_z\right)\right]|\psi\rangle,$$

where instead the formula from exercise **1c** we use instead the representation

$$\vec{v} = \begin{pmatrix} \langle\psi|\sigma'_x|\psi\rangle \\ \left\langle\psi\middle|\sigma'_y\middle|\psi\right\rangle \\ \langle\psi|\sigma'_z|\psi\rangle \end{pmatrix},$$

where σ'_x, σ'_y, and σ'_z are the Pauli matrices in the $|\pm u\rangle$ basis, given by

$$\sigma'_x = |+u\rangle\langle -u| + |-u\rangle\langle +u|, \quad \sigma'_y = -i|+u\rangle\langle -u| + i|-u\rangle\langle +u|, \quad \sigma'_z = |+u\rangle\langle +u| - |-u\rangle\langle -u|$$

Plot the result as a function of t for some specific $|\psi\rangle \neq |\pm u\rangle$, and describe it with words (you don't need to print the graph!)

Hinweis: Don't panic! Compute first $\left(a\sigma_x + b\sigma_y + c\sigma_z\right)^2$ and simplify it as much as possible.