Quantenmechanik

David Gross, Mateus Araújo

Sheet 3 **Due: 23.04 um 12 Uhr**

1 Unitarity (6 P)

An operator U is *unitary* if $U^\dagger U = UU^\dagger = \mathbb{1}$.
In this exercise we are going to introduce fundamental properties of unitary operators and show that the set of unitary operators coincides with the set of quantum mechanical time evolution operators.

a) **(0,5 P)** Let $|\psi\rangle$ be an arbitrary quantum state, where $\||\psi\rangle\|_2 = 1$, and U an arbitrary unitary operator. Show that

$$\|U|\psi\rangle\|_2 = 1.$$

This means that unitary time evolution preserves the normalisation of quantum state.

b) **(0,5 P)** Let $\{|\psi_i\rangle\}_{i=1}^d$ and $\{|\varphi_i\rangle\}_{i=1}^d$ be two arbitrary orthonormal bases. Show that the operator $U = \sum_{i=1}^d |\psi_i\rangle\langle\varphi_i|$ is unitary

c) **(1 P)** Let

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be an arbitary unitary operator with dimension $d = 2$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Find the conditions that $\alpha, \beta, \gamma, \delta$ need to fulfill so that U is in fact unitary. Find also a specific choice of $\alpha, \beta, \gamma, \delta$ that fulfills these conditions and additionally $|\alpha| = |\gamma|$ (the solution is not unique).

d) **(1,5 P)** Show that any unitary operator V can be written as $\sum_i |\gamma_i\rangle\langle i|$ and $\sum_i |i\rangle\langle\delta_i|$, where $\{|i\rangle\}_i$ is the standard basis, and $\{|\gamma_i\rangle\}$ and $\{|\delta_i\rangle\}$ orthonormal bases. (This means that the rows and columns of every unitary matrix are orthonormal bases).

e) **(1 P)** Let $U = \exp(itH)$, where $t \in \mathbb{R}$ and $H = H^\dagger$. (U is the time evolution operator obtained from the Hamiltonian H). Show that $U^\dagger = \exp(-itH)$, and that U is unitary.

f) **(1,5 P)** The spectral theorem says that if a finite-dimensional operator M is normal – that is if $[M, M^\dagger] = 0$ – then M can be written as

$$M = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|,$$

where $\lambda_i \in \mathbb{C}$ and $\{|\psi_i\rangle\}$ is an orthonormal basis. Show that every unitary operator U is normal, and that its eigenvalues can be written as $e^{i\theta}$, with $\theta \in \mathbb{R}$. With that, show that every unitary operator U can be written as $U = \exp(iH)$, where H is a self-adjoint operator.

2 Measurements (4 P)

A von Neumann measurement of quantum state $|\psi\rangle$ of dimension $d = 2$ with results $\heartsuit$ and $\spadesuit$ is described by two projectors $\{\Pi_\heartsuit, \Pi_\spadesuit\}$ such that $\Pi_\heartsuit + \Pi_\spadesuit = \mathbb{1}$. Result $\heartsuit$ happens with probability $p(\heartsuit) = \|\Pi_\heartsuit|\psi\rangle\|_2^2$, and leaves the quantum system in the state $\Pi_\heartsuit|\psi\rangle / \|\Pi_\heartsuit|\psi\rangle\|_2$. Result $\spadesuit$ happens with probability $p(\spadesuit) = \|\Pi_\spadesuit|\psi\rangle\|_2^2$, and leaves the quantum system in the state $\Pi_\spadesuit|\psi\rangle / \|\Pi_\spadesuit|\psi\rangle\|_2$.

a) **(0,5 P)** Show that for any quantum state $|\psi\rangle$ of dimension 2, and any projectors such that $\Pi_{\heartsuit} + \Pi_{\spadesuit} = \mathbb{1}$ we have that $p(\heartsuit) \geq 0, p(\spadesuit) \geq 0$, and $p(\heartsuit) + p(\spadesuit) = 1$.

b) **(0,5 P)** Show that for any 2-dimensional quantum state $|\theta\rangle = \gamma|0\rangle + \delta|1\rangle$, with $\gamma, \delta \in \mathbb{C}$ and $|\gamma|^2 + |\delta|^2 = 1$, the "ketbra" $|\theta\rangle\langle\theta|$ is a projector.

Reminder: A (orthogonal) projector is an operator Π such that $\Pi^\dagger = \Pi$ and $\Pi^2 = \Pi$.

c) **(1 P)** Let $\Pi_{\heartsuit} = |\theta\rangle\langle\theta|$ be the projector from exercise **2b**. Find $\Pi_{\spadesuit}$ such that $\{\Pi_{\heartsuit}, \Pi_{\spadesuit}\}$ is a von Neumann measurement. Find a 2-dimensional quantum state $|\theta^\perp\rangle$ such that $\Pi_{\spadesuit} = |\theta^\perp\rangle\langle\theta^\perp|$ (the solution is not unique).

d) **(1 P)** Write down explicitly the two possible quantum states one can obtain after making the von Neumann measurement $\{\Pi_{\heartsuit}, \Pi_{\spadesuit}\}$ from exercise **2c** on a 2-dimensional quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

e) **(1 P)** It is often the case that an experimental apparatus can only do measurements in a fixed basis $\{|0\rangle, |1\rangle\}$, with results 0, 1, and projectors $\Pi_0 = |0\rangle\langle 0|$ and $\Pi_1 = |1\rangle\langle 1|$. Measurements in other bases are implemented by applying some unitary U to the state before measuring it in this fixed basis. Find a unitary U such that $p(0) = \|\Pi_0 U|\psi\rangle\|_2^2 = \|\Pi_{\heartsuit}|\psi\rangle\|_2^2$ and $p(1) = \|\Pi_1 U|\psi\rangle\|_2^2 = \|\Pi_{\spadesuit}|\psi\rangle\|_2^2$ for all states $|\psi\rangle$ (the solution is not unique). What are the post-measurement states?