Quantenmechanik

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Sheet 5 Due: 07.05 um 12 Uhr

1 One-dimensional tunneling (10 P)

Consider the one-dimensional rectangular potential barrier

$$V(x) = \begin{cases} 0 & : \; x \leq 0 \\ V_0 & : \; 0 < x < d \\ 0 & : \; x \geq d \end{cases} \qquad \text{mit } \; V_0 > 0\,.$$

a) (2 P) The Schrödinger equation for this system is

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \right)\psi(x,t).$$

We consider the case $E < V_0$. Show that the function $\psi(x,t) = e^{-\frac{i}{\hbar}Et}\varphi(x)$ with

$$\varphi(x) = \begin{cases} Ae^{iax} & + & Be^{-iax} & : & x \leq 0 \\ Ce^{\alpha x} & + & De^{-\alpha x} & : & 0 < x < d \\ Fe^{iax} & + & Ge^{-iax} & : & x \geq d \end{cases} \qquad \text{with } a = \frac{\sqrt{2mE}}{\hbar}, \quad \alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

is a solution of the Schrödinger equation for this potential.

b) (2 P) Assume that $\varphi(x)$ and $\varphi(x)'$ are continuous. Show that this assumption leads to the follow system of linear equations:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 - \frac{i\alpha}{a} & 1 + \frac{i\alpha}{a} \\ 1 + \frac{i\alpha}{a} & 1 - \frac{i\alpha}{a} \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix},$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1 + \frac{ia}{\alpha})e^{-\alpha d + iad} & (1 - \frac{ia}{\alpha})e^{-\alpha d - iad} \\ (1 - \frac{ia}{\alpha})e^{\alpha d + iad} & (1 + \frac{ia}{\alpha})e^{\alpha d - iad} \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}.$$

c) (2 P) We consider the case $G = 0$. In the lecture we shall see that in this case we can interpret A as the amplitude of a particle coming from $x = -\infty$, and F as the amplitude of particle going to $x = +\infty$. Therefore we call $T(E) = \frac{F}{A}$ the *transmission amplitude*. Show that the *transmission coefficient* $|T(E)|^2$ is given by

$$|T(E)|^2 = \frac{1}{1 + \left(1 + \frac{1}{4}\left(\frac{\alpha}{a} - \frac{a}{\alpha}\right)^2\right)\sinh^2 \alpha d}\,.$$

d) (1 P) Sketch $|T(E)|^2$, preferably with a computer. Choose for example $\hbar = 1, m = 1/2, V_0 = 10, d = 1$. How does the transmission coefficient behaves as a function of the energy? (No points, but also interesting: Sketch the behaviour of the transmission coefficient for a fixed energy as a function of the width of the barrier d).

e) (3 P) Here we shall visualise the wave function $\varphi(x)$. For that, assume that $G = 0, F = 1, m = 9.1 \times 10^{-31}$ kg, and $\hbar = 6.63 \times 10^{-34}$ J s. Choose values for E, V_0, and d based on the results of item **1d** so that one can clearly see the tunneling. These parameters, together with the result of item **1b**, determine the wave function $\varphi(x)$ completely. Sketch (again preferably with a computer) the real part, the imaginary part, and the absolute value squared of $\varphi(x)$. What is the physical meaning of the fact that the frequency before and after the barrier is different, whereas the amplitude differs?

2 (Bonus exercise) Beam me up, Scotty (2 P)

The gravitational potential of the "surface" of Uranus and Neptune is approximately equal (https://xkcd.com/681), and so we can roughly model this system with the rectangular potential of Exercise 1. Consider then that during a conjunction between Neptune, Uranus, and the Sun, an astronaut flies away from the "surface" of Uranus using their jetpack with a velocity of 20 m/s in the direction of Neptune. Compute the values of E, V_0 and d for this situation under the assumption that the mass of the astronaut (with jetpack) is 200 kg, and with that the probability that the astronaut tunnels to Neptune. The answer only needs to be correct in order of magnitude, but zero is not an answer!