Quantenmechanik

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Sheet 6 **Due: 14.05 um 12 Uhr**

1 The free wavepacket (10 P)

In this exercise we shall calculate the time evolution of the Gaußian wavepacket.

a) (1 P) Show that when $V(x) = 0$ the general solution of the Schrödinger equation has the form

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k\, g(k) e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)},$$

where $g(k)$ is a normalised function (i.e., $\int_{-\infty}^{\infty} \mathrm{d}k\, |g(k)|^2 = 1$).

b) (1 P) Consider a wavepacket that at time $t = 0$ is given by

$$\Psi(x,0) = A e^{ik_0 x} e^{-x^2/a^2}.$$

Show that the normalisation constant A is equal to $\sqrt[4]{\frac{2}{\pi a^2}}$.

Hint: For that you need calculate the famous Gaußian integral $\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-px^2} = \sqrt{\pi/p}$. It is easy when one writes $\left(\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-px^2}\right)^2$ as a double integral over the $x-y$ plane, and changes from rectangular to polar coordinates. Fact for later (you don't need prove this): this result is also true for complex p with $\Re(p) > 0$.

c) (1 P) To solve Gaußian integrals with complex numbers, the fact that

$$f(\alpha,\beta) = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-p(x+\alpha+i\beta)^2}$$

does not depend on α and β is quite helpful. Here we assume that α and β are real, and p is complex with $\Re(p) > 0$. To see that, first show that

$$\frac{\partial}{\partial \beta} f(\alpha,\beta) = i \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\partial}{\partial x} e^{-p(x+\alpha+i\beta)^2} = 0.$$

Show also that $f(\alpha,\beta)$ does not depend on α. Compute $f(\alpha,\beta)$.

d) (2 P) Show that the function $g(k)$ from item **1a** is given by

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \Psi(x,0) e^{-ikx}.$$

Choose $\Psi(x,0)$ as in item **1b** and show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \Psi(x,0) e^{-ikx} = \sqrt[4]{\frac{a^2}{2\pi}} e^{-\frac{a^2}{4}(k-k_0)^2}.$$

Hint: Solve this integral using items **1b** and **1c**. Alternatively, you can use item **2c** from Sheet 4 together with the fact that the Fourier transform doesn't change the L^2 norm of a function.

e) **(2 P)** The solution for a wave packet with Gaußian initial state is therefore

$$\Psi(x,t) = \frac{\sqrt{a}}{(2\pi)^{\frac{3}{4}}} \int_{-\infty}^{\infty} \mathrm{d}k \, e^{-\frac{a^2}{4}(k-k_0)^2} e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}.$$

Show that

$$\Psi(x,t) = \left(\frac{2a^2}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{a^2 + \frac{2i\hbar t}{m}}} \exp\left(\frac{-x^2 + ia^2k_0x - ia^2k_0^2t\hbar/2m}{a^2 + \frac{2i\hbar t}{m}}\right)$$

Hint: Complete the square to bring the integral to the form used in item **1c**.

f) **(2 P)** Show that the probability density $|\Psi(x,t)|^2$ is given by

$$|\Psi(x,t)|^2 = \left(\frac{2a^2}{\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{a^4 + \frac{4\hbar^2t^2}{m^2}}} \exp\left(\frac{-2a^2(x - \frac{\hbar k_0}{m}t)^2}{a^4 + \frac{4\hbar^2t^2}{m^2}}\right).$$

Where is the maximum of $|\Psi(x,t)|^2$ (as function of x), and how does it move (as function of t)?
Hint: $|e^z|^2 = e^{2\Re(z)}$

g) **(1 P)** The standard deviation of an operator A is given by $\Delta A(t) = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$. Show that for $\Psi(x,t)$ and the position and momentum operators the standard deviations are given by

$$\Delta x(t) = \frac{a}{2}\sqrt{1 + \frac{4\hbar^2t^2}{m^2a^4}} \quad \text{und} \quad \Delta p(t) = \frac{\hbar}{a}.$$

Show that these values are consistent with the uncertainty relation $\Delta x \Delta p \geq \hbar/2$.

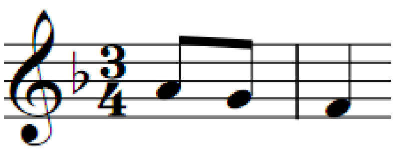
Hint: $|\Psi(x,t)|^2$ is just a Gaußian distribution...

2 (Bonus exercise) Time-frequency uncertainty (2 P)

The Heisenberg uncertainty relation $\Delta x \Delta p \geq \hbar/2$ is also relevant for classical signals. Consider an acoustic signal $\psi(t)$, where t is th elapsed time in seconds. The time operator $(T\psi) = t\psi(t)$ measures the centre of the signal in the time axis: $\langle\psi|T|\psi\rangle = \int_{-\infty}^{\infty} t\psi(t)^2$. Analogously the frequency operator measures $(F\psi)(t) = -i\frac{1}{2\pi}\frac{\partial}{\partial t}\psi(t)$ the frequency of ψ.

From the commutation relation $[x,p] = i\hbar$ for the position and momentum operators, derive the commutation relation for T and F and from that prove the time-frequency uncertainty relation $\Delta t \Delta f \geq 1/(4\pi)$.

The notes below code information about time and frequency simultaneously. By doing that they could contradict the uncertainty relations. Estimate Δt and Δf roughly for these notes, and argue that here no problems arises.



The lowest note that a piano can play is an A with frequency 27.5 Hz. Is it possible to play this note as a 32nd (demisemiquaver) with tempo 120 bpm?