Quantenmechanik

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Sheet 9 **Due: 04.06 um 12 Uhr**

1 Spin-1 Operators (4 P)

In the lecture we have derived a matrix represenation for the spin-1/2 operators. Here we shall work out the other interesting case: spin-1. It is given that

$$S^2|s,m\rangle = s(s+1)|s,m\rangle$$
$$S_z|s,m\rangle = m|s,m\rangle$$
$$S_+|s,m\rangle = \sqrt{(s(s+1)-m(m+1))}|s,m+1\rangle$$

a) (0,5 P) Show that in the base $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ the operator S_z is represented as

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

b) (0,8 P) Consider the matrix elements $\langle 1,m|S_+|1,m'\rangle$. For which values of m, m' can the elements be different than zero? With that, show that in this basis the raising operator is represented as

$$S_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

c) (0,7 P) Using the result of **1b** and the fact that $S_+^\dagger = S_-$, show that the operators S_x and S_y are represented as

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{und} \quad S_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

d) (2 P) Show that the operators S_x^2, S_y^2, and S_z^2 commute. Find the basis in which all three operators are diagonal, and write down the operators S_x, S_y, and S_z in this basis.

2 Turn around and change the world (6 P)

We want to understand a surprising quantum mechanical effect: if we rotate a spin-1/2 by 360 degrees, it does *not* come back to the original state.

Remember exercises **2a** and **2b** from sheet 2 (this is a good opportunity for repetition!). Then we have shown that

$$U(\theta) = \exp(i\theta\sigma_x/2) = \cos(\theta/2)\mathbb{1} + i\sin(\theta/2)\sigma_x,$$

describes a rotation of θ around the x axis (and not of $\theta/2$). One can see immediately that $U(2\pi) = -\mathbb{1}$. Therefore, the effect of $U(2\pi)$ is not observable, when the operator is applied to the *full* wavefunction (Sheet 2, **1b**). Below we shall add a further degree of freedom, that makes it so that the phase is only applied to a part of the wavefunction and becomes therefore observable.

a) **(1,5 P)** The system we shall model consists of a spin-1/2 particle with one additional degree of freedom: it can be in one of two places. We denote them as "1" and "2". The Hilbert space $\mathcal{H}$ of this particle is spanned by four vectors:

$$|\uparrow, 1\rangle, |\uparrow, 2\rangle, |\downarrow, 1\rangle, |\downarrow, 2\rangle.$$

They are: "Spin up, place 1", "Spin up, place 2", and so on.

We introduce a unitary operation H that puts the particle in a superposition of both places. It acts on the position degree of freedom as

$$H|1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \qquad H|2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle).$$

and leaves the spin unchanged. (Such an H is called a *Beamsplitter*). To work out the action of H on the whole basis, we write down the spin and position degrees of freedom as a product $|\uparrow, 1\rangle = |\uparrow\rangle|1\rangle$, apply H to the position degree of freedom and multiply:

$$H|\uparrow, 1\rangle = |\uparrow\rangle(H|1\rangle) = |\uparrow\rangle\Big(\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\Big) = \frac{1}{\sqrt{2}}\Big(|\uparrow, 1\rangle + |\uparrow, 2\rangle\Big).$$

What is the matrix representation of H in the basis of $\mathcal{H}$ given above?

b) **(1,5 P)** Let $R(\theta)$ be the operation that applies a θ rotation to the spin if the particle is in place "1". If the particle is in place "2", it stays unchanged. Therefore z.B.

$$R(\theta)(|\uparrow\rangle|1\rangle) = (U(\theta)|\uparrow\rangle)|1\rangle, \qquad R(\theta)(|\uparrow\rangle|2\rangle) = |\uparrow\rangle|2\rangle.$$

Write down the matrix representation of $R(\theta)$.

c) **(3 P)** Consider now the following experiment: The particle starts in place "1" with spin up. We apply the operation H to put the particle in a superposition of both places. Then we apply $R(\theta)$, a rotation $U(\theta)$ on place "1". To conclude we apply H again.

Compute the state of the system after the first, second, and third steps of the experiment. Show that at the end of the experiment the probability that the particle is observed again with spin up is

$$\frac{1}{4}\big(\cos\theta/2 + 1\big)^2.$$

On how many degrees must one rotate so that the initial quantum state is recovered?

This calculation models an experiment that was done with neutrons in the 70s. The title of the publication is Verification of coherent spinor rotation of Fermions (linked here). You can access it from the University Network.