Classical Mechanics

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Exercise sheet 13 **Due: January, 25 at 12:00**

1 Rolling mill

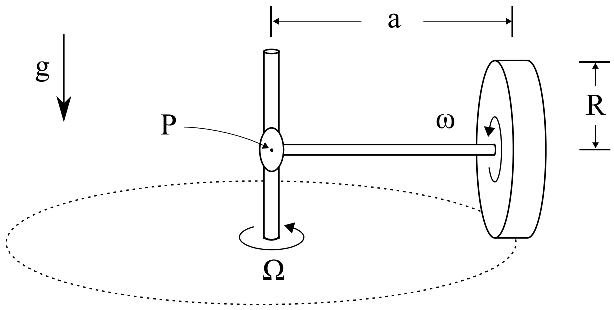

Figure 1: A millstone of radius R is forced to roll in a circle of radius a by a shaft that rotates at the angular frequency Ω. The millstone rolls without slipping, resulting in an angular frequency ω around the axis of the millstone.

In a rolling mill, a disk shaped millstone rolls in a circle on a flat surface, driven by a vertical shaft. Clearly, the weight of the millstone contributes to a contact force that crushes the grains. What maybe is less obvious is that the contact force can be considerably larger than that resulting from the mere weight of the wheel. Here we shall try to understand why this is the case.

We assume that millstone has weight M and radius R (one does not need to know the thickness of the millstone for what we are going to do). The millstone is supported on an axis. The axis is in turn attached to a vertical shaft by a joint P. The length of the axis from the point P to the center of mass of the millstone is a, and we assume that we can ignore the mass of the axis. We also assume that the millstone rolls without slipping. The shaft rotates with the angular frequency Ω, and the millstone rotates around its symmetry axis with the angular frequency ω.

a) What is the relation between Ω and ω?

Hint: This follows from the condition that the millstone rolls without slipping.

(1 points)

b) The rotation ω corresponds to an angular momentum vector $\vec{L}_{\parallel}$. *What is $\vec{L}_{\parallel}$, and how does it depend on time?*

Hint: What is the angular velocity vector $\vec{\omega}$ that describes the rotation of the millstone around its symmetry axis, and how does that depend on time? How does this translate to an angular momentum vector?

Remark: $\vec{L}_{\parallel}$ is not the total angular momentum of the system. With respect to the point P there is also a vertical component, due to Ω. Luckily (it would have been a more complicated problem), we not have to know that component.

(3 points)

c) *What torque (with respect to P) is required in order to yield the change of the angular momentum that you obtained in b)?*

(2 points)

d) The only forces that can result in the torque in c) is the gravitational force, and the normal force from the ground, acting on the wheel. *How large does the normal force have to be in order to give the correct torque? At what value of Ω is the normal force twice that for the millstone being at rest?*

(2 points)

2 Inertia tensor of object Å

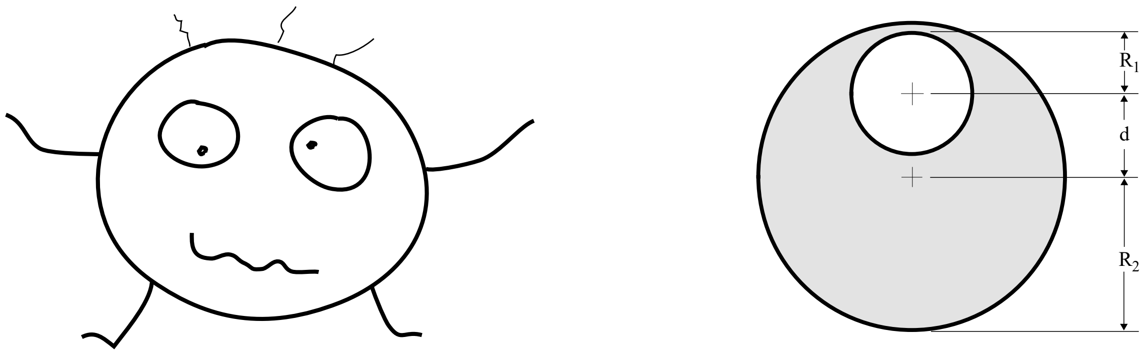

Figure 2: To the left is displayed a realistic representation of object Å. This body has total mass M, and is accurately approximated by a sphere of radius R_2 with uniform density, wherein there is a spherically shaped cavity (up towards the head) containing absolute vacuum. This spherical cavity has radius R_1, and its center is displaced by a distance d from the center of the large sphere. We have $R_2 \geq R_1 + d$, i.e., the cavity lies completely within the larger sphere.

a) *How are the principal axes of object Å oriented?* **(1 points)**

b) *Determine the inertia tensor, with respect to a coordinate system with origin at the center of the big sphere, and oriented along the principal axes.*

(4 points)

3 Intermediate axis theorem (Tennis racket theorem)

An asymmetric top is a rigid body where all three principal moments are different $I_1 > I_2 > I_3 > 0$[1]. That the top is free means that it is not affected by any torque. This means that the Euler equations, in a body-fixed coordinate system[2] that coincides with the principal axes reduce to

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 &= 0, \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0, \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= 0, \end{aligned} \tag{1}$$

where ω_1, ω_2, and ω_3 are the components of the angular velocity vector with respect to the principal body-fixed coordinate system.

a) Suppose that the rotation initially is almost perfectly aligned with the 1-axis, i.e. $\omega_1 \gg \omega_2$ and $\omega_1 \gg \omega_3$. As an approximation we put $\omega_3\omega_2$ to zero in the first line of (1). *Solve* (1) *under this approximation.*

[1] A tennis racket would typically have different principal moments of inertia.

[2] That it is body-fixed means that the coordinate system is 'glued' onto the body, i.e., it follows the motion of the body. One should keep in mind that this generally is a non-inertial coordinate system.

Find the analogous approximate solutions for the the case $\omega_2 \gg \omega_1$, $\omega_2 \gg \omega_3$ as well as the case $\omega_3 \gg \omega_1$, $\omega_3 \gg \omega_2$. **(5 points)**

b) What do the results in a) suggest concerning the stability of the rotation around the three principal axes of an asymmetric top? **(2 points)**