Classical Mechanics

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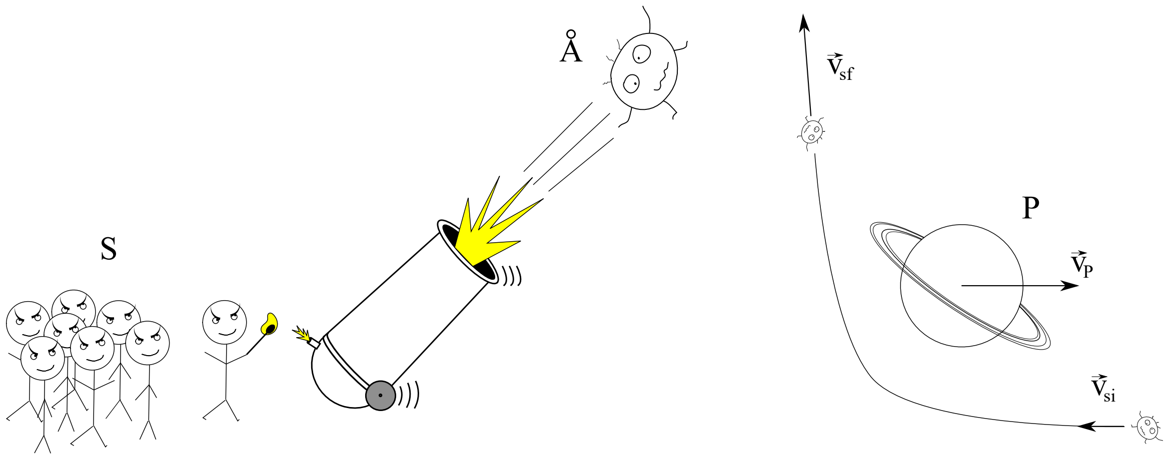
Exercise sheet 3 **Due: November, 2 at 12:00**

1 Shooting something very far into space



A group of students S want to send an individual Å, who creates exercises in classical mechanics, very far into space. They use a cannon to shoot this object into space[1]. They have enough gunpowder to reach the necessary escape velocity from Earth. However, they want to make sure that Å disappears deeply into interstellar space, but at the same time they want to use as little gunpowder as possible, so that they have more money left for the party afterwards. Some of them have read a wikipedia-article about a clever technique called the "gravitational slingshot´´ (or "gravity assist maneuver´´, or "swing-by´´). The idea is that one can increase the speed (relative to the sun) of a spacecraft (or an object Å) by letting it pass near a planet in a suitable way. The problem is that the group of students S are not quite sure how all of this works, so you have to help them to understand the general principles.

As mentioned above, the gravitational sling-shot works by letting object Å pass near a planet P. For the first step of the analysis we consider the system in the center of mass frame of Å and P[2]. We assume that the mass M of P is much larger than the mass m of Å. Hence, we can to a very good approximation put the center of mass at the center of P, and the relative mass μ equal to m. The total energy of Å is positive[3] since otherwise Å would be trapped in a periodic orbit around P[4]. This means that it follows a hyperbolic orbit, in other words, $r(\phi) = p/(1+\epsilon\cos\phi)$, where the eccentricity is such that $\epsilon > 1$. Very far away from P, object Å does to a good approximation only have kinetic energy, and approximately follows a straight line. The minimal distance between P and this line is often called the impact parameter. We let $\vec{v}_{pi}$ and $\vec{v}_{pf}$ denote the initial and final velocity of Å very far from P, in the center of mass frame.

a) *Show that the initial speed $v_{pi} = \|\vec{v}_{pi}\|$ and final speed $v_{pf} = \|\vec{v}_{pf}\|$ are equal.* **(1 points)**

b) *For the center of mass frame, express the total energy E and the angular momentum L in terms of the mass m, the initial speed v_{pi}, and the impact parameter b.* **(2 points)**

[1] Although they do intend to become dictators (as one can see on their eyebrows) they are still only doing dictator-internships, so they can unfortunately not afford a rocket.

[2] We ignore the influence from the sun.

[3] We follow the convention to put the potential energy to zero infinitely far away from P.

[4] Strictly speaking zero energy would also be allowed. Then, the orbit would be parabolic (and thus $\epsilon = 1$), but we skip that special case here.

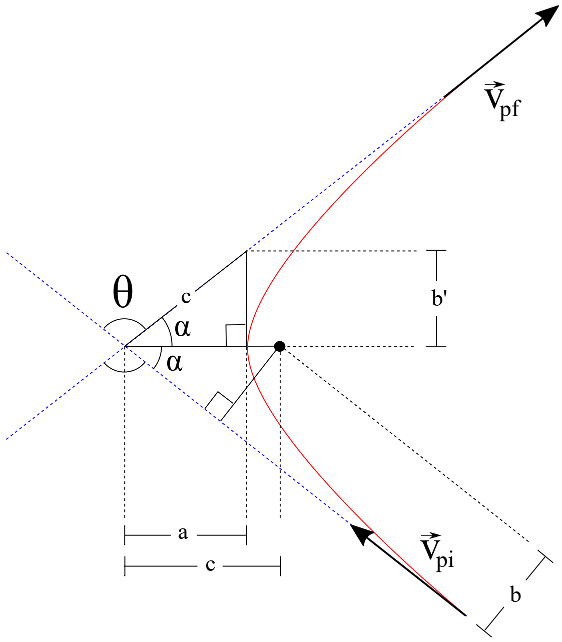


Figure 1: The horizontal axis is the symmetry axis of the hyperbola (in red). The dotted blue lines correspond to the asymptotes of the hyperbola, which object Å approximately travels along when far from P. There are two distances that are marked as equal in this figure, with the value c. This is due to the properties of the hyperbola, and can be used to show that $b' = b$.

c) *Use simple geometric arguments to show that the length b', as shown in figure 1, is equal to the impact parameter b.* **(2 points)**

d) The scattering angle θ is defined as the angle by which the initial velocity $\vec{v}_{pi}$ is turned into $\vec{v}_{pf}$. (Hence, we do for example have $\vec{v}_{pi} \cdot \vec{v}_{pf} = v_{pi} v_{pf} \cos\theta$.) *Derive the scattering angle θ as a function of m, v_0 and b.*

Hint: Recall that the eccentricity ϵ of the orbit is given by the ratio of two characteristic lengths, $\varepsilon = \frac{c}{a}$, shown in figure 1. Use this to show that $\epsilon = \sqrt{1 + \tan^2\alpha}$. Next, recall the relation between ϵ and L, E, and m, and use exercise b). Use this to find an expression for the scattering angle θ. **(3 points)**

e) The ultimate goal is to determine the difference between the final kinetic energy $E_{\text{kin},i}$ of object Å and and the initial kinetic energy $E_{\text{kin},f}$, *with respect to the reference frame of the sun*. Let $\vec{v}_P$ denote the velocity of planet P (also in the frame of the sun). *Show that*

$$E_{\text{kin},f} - E_{\text{kin},i} = m(\vec{v}_{pf} - \vec{v}_{pi}) \cdot \vec{v}_P.$$

Hint: Let $\vec{v}_{si}$ and $\vec{v}_{sf}$ be the initial and final velocities of Å *in the reference frame of the sun*. What is the relation between $\vec{v}_{si}$ and $\vec{v}_{sf}$ and $\vec{v}_{pi}$ and $\vec{v}_{pf}$? What is the relation between $E_{\text{kin},i}$ and $E_{\text{kin},f}$ and $\vec{v}_{si}$ and $\vec{v}_{sf}$? Use what you know from a). **(2 points)**

f) To demonstrate the gravitational slingshot one can consider the special case that the initial velocity $\vec{v}_{si}$ of Å, in the frame of the sun, is anti-parallel to $\vec{v}_P$. In other words, Å initially moves in the opposite direction compared to planet P. Hence, we let $\vec{v}_{si} = -s\vec{v}_P$ for $s > 0$. *Show that*

$$E_{\text{kin},f} - E_{\text{kin},i} = m(1+s)\|\vec{v}_P\|^2\Big(1 - \cos\theta\Big),$$

where θ is the scattering angle that we discussed in d). What does this imply for the relation between $E_{\text{kin},f}$ and $E_{\text{kin},i}$? Where does the energy come from? **(3 points)**

2 Barge on a river

Imagine that you are operating a barge that is attached to huge ropes that stretch over a river. By pulling the ropes you can move the barge around. The flow of the river exerts a force on the barge which we will assume to be $\vec{F} = q\vec{v}$, where $\vec{v}$ is the velocity of the flow and $q > 0$ is some constant.

If the barge is pulled very slowly around on the river, we can assume that all of the force is directly transferred to the machinery that pulls the ropes. Suppose that the velocity profile of the current is $\vec{v} = v_0(1 - \frac{x^2}{a^2})\hat{y}$ for $-a \leq x \leq a$ and $v_0 > 0$. Here, $x = -a$ and $x = a$ are the two banks of the river and $\hat{y}$ denotes the unit vector in the positive y-direction. Hence, we have oriented the coordinate system such that the positive y-direction is in the direction of the flow of river, the x-axis is perpendicular to the river, and $x = 0$ is in the middle.

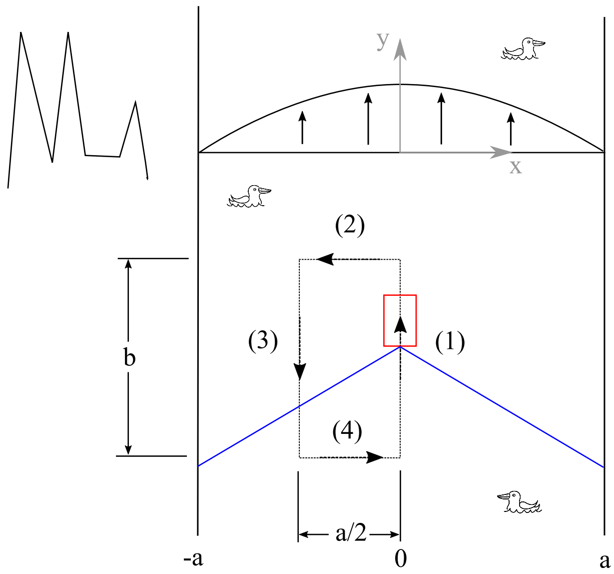


Figure 2: A barge (red) is attached to two huge ropes (blue) that stretch over a river and is moved along the indicated cycle mentioned in the text.

Next, let us assume that we want to move the barge along a closed cycle C as indicated in Fig. 2. The cycle consists of four straight lines described by the following sequence of points $(x, y) = (0, 0) \rightarrow (0, b) \rightarrow (-a/2, b) \rightarrow (-a/2, 0) \rightarrow (0, 0)$, where $a, b > 0$.

a) *Evaluate the curl $\vec{\nabla} \times \vec{F}$ in Cartesian coordinates. Is the force field conservative or non-conservative?* Think of the force field as being independent of z. **(2 points)**

b) *By evaluating the line integral of the force exerted by the river on the barge, determine how much energy that you would gain, or need to spend, in order to pull the barge once around the cycle C. Where does that energy come from?* **(3 points)**

c) *Confirm the result in (b) by using Stokes theorem.*

Hint: Recall that Stokes theorem says that $\oint_C \vec{F} \cdot d\vec{r} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A}$, where the right hand side is the surface integral over an area S enclosed by the closed loop C on the left hand side[5]. Keep in mind that the orientations of the loop C and the surface S have to obey the right-hand rule.

(3 points)

[5] I. e. $C = \partial S$