Classical Mechanics

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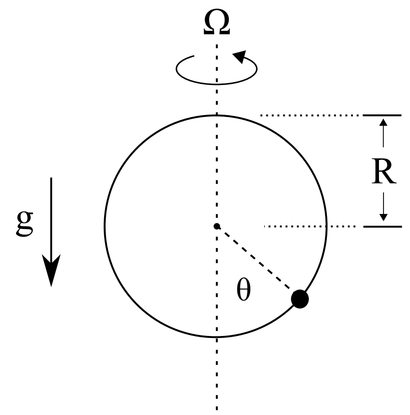
Exercise sheet 6 **Due: November, 23 at 12:00**

1 Bead on a rotating hoop



A bead of mass m slides without friction on a hoop of radius R that rotates around its vertical axis with a constant angular speed Ω.

a) *Derive the Lagrangian expressed in terms of the angle θ that the bead makes with the vertical downwards direction.* **(2 points)**

b) The Lagrangian that you obtain is equivalent to that of a particle of mass M that moves along the real line, and being affected by a (periodic) potential $V(\theta)$. *Determine M and $V(\theta)$.*

(1 points)

c) *Find the Euler-Lagrange equation of the bead.* **(2 points)**

d) The equilibrium configurations of the bead depend on the angular speed Ω. *As a function of Ω, determine the equilibrium configurations, and show whether they are stable or unstable.*

Hint: You will find different regimes depending on Ω.

(4 points)

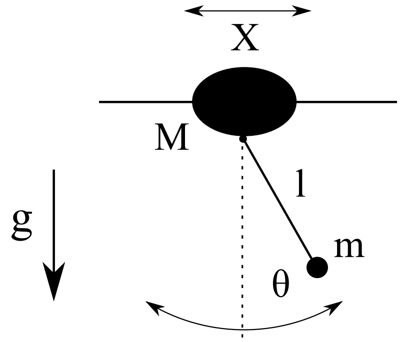
2 Pendulum on a sliding support

An object of mass M can slide without friction along a horizontal rail. From the object hangs a massless rod of length l with a mass m attached to the end. The rod can swing (without friction) in the plane along the rail and the vertical direction.

a) *Derive the Lagrangian in terms of the position X of the mass M, and the angle θ.* **(3 points)**

b) *Derive the Euler-Lagrange equations.* **(3 points)**

c) *Find the approximate equation in the the small angle approximation.* This means that we assume that θ is so small that we can make the approximation $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$, and that we in the resulting equations only keep terms up to first order in θ, $\dot{\theta}$, and $\ddot{\theta}$ (e.g. θ^2 and $\theta\dot{\theta}$ are second order terms, while $\theta\dot{\theta}^2$ is a third order term). **(2 points)**



d) *Find the complete solution to the approximate equations of motion in c). What are the normal modes, and the corresponding frequencies?* **(3 points)**

3 Generalized Euler-Lagrange equation

This problem gives no points, but a gold star. If you feel uncertain about the theory behind the Euler-Lagrange equations, then this is the exercise for you!

Consider the functional

$$\mathcal{S}[x] = \int_{t_i}^{t_f} f(t, x, \dot{x}, \ddot{x}) dt,$$

i.e., compared to the standard case, we have the additional dependence on $\ddot{x}$.

Show that if $x(t)$ is a stationary point of $\mathcal{S}[x]$, then it satisfies

$$\frac{d^2}{dt^2}\left(\frac{\partial f}{\partial \ddot{x}}\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\partial f}{\partial x} = 0.$$

This requires some boundary conditions. What are these boundary conditions?

Hint: The standard E-L equation can be obtained by using the boundary conditions that the values of $x(t_i)$ and $x(t_f)$ are fixed. How could this be generalized? Look up how the derivation of the standard Euler-Lagrange equations works, and generalize that derivation.

(0 points, but a gold star!)