Classical Mechanics

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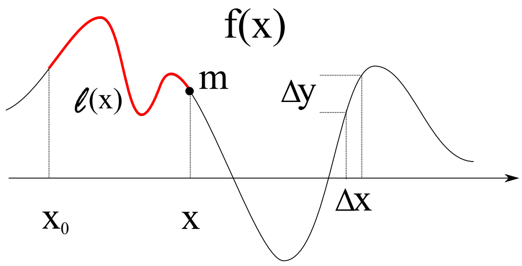
Exercise sheet 7 **Due: November, 23 at 12:00**

1 Change of variable for a constrained particle

Imagine a particle that is constrained to move along a curve (x, y) where $y = f(x)$[1]. The particle is otherwise not affected by any potentials.



a) *What is the Lagrangian of this system, in terms of the coordinate x?* **(1 points)**

b) *Derive the Euler-Lagrange equation.* **(2 points)**

c) *Find the general expression for the length $\ell(x)$ of the curve, from the point $(x_0, f(x_0))$ to $(x, f(x))$, for some arbitrary but fixed x_0.*

Hint: The answer is an integral. Think of the interval $[x, x + \Delta x]$ for a small Δx. What is the approximate size of Δy? What is the length of the line that connects the point (x, y) to $(x + \Delta x, y + \Delta y)$? **(2 points)**

d) *Make a change of variables to the new coordinate ℓ in the Lagrangian. What is the Euler-Lagrange equation in this case? Find the general solution.* **(2 points)**

2 Fictitious forces

We can change variables in the Lagrangian to any new coordinates that we like, even time-dependent non-inertial ones. The result is always a valid Lagrangian for the system. Consider a point mass m in free space. In a Cartesian inertial coordinate system (x, y, z) this particle has the very simple Lagrangian $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Suppose now that we would like to describe the motion of this particle with respect to a new coordinate system (x', y', z') that rotates with uniform angular speed ω around the z-axis. In other words

$$\begin{aligned} x' &= x\cos(\omega t) + y\sin(\omega t), \\ y' &= -x\sin(\omega t) + y\cos(\omega t), \\ z' &= z, \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} x &= x'\cos(\omega t) - y'\sin(\omega t), \\ y &= x'\sin(\omega t) + y'\cos(\omega t), \\ z &= z'. \end{aligned}$$

a) *Determine the Lagrangian and the Euler-Lagrange equations in the new coordinate system (x', y', z').*

(2 points)

[1] f is a nice and smooth function.

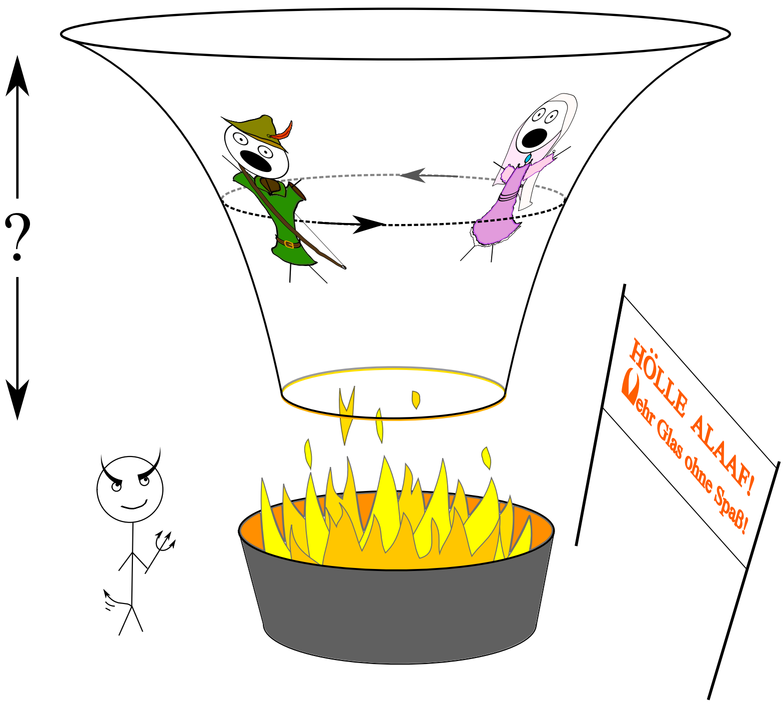
b) Let $\vec{\omega} = (0, 0, \omega)$ and $\vec{r}' = (x', y', z')$. *Show that the Euler-Lagrange equations in a) can be rewritten as*

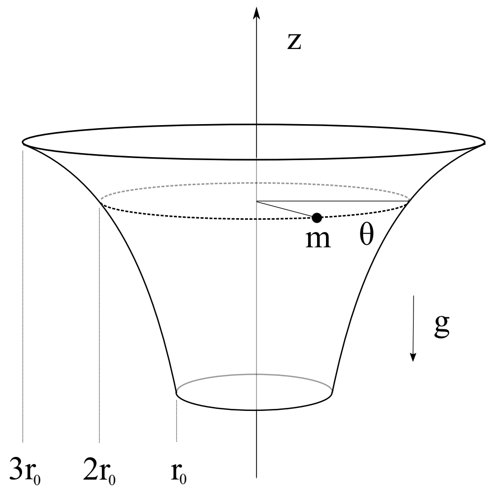
$$m\ddot{\vec{r}}' = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}') - 2m\vec{\omega} \times \dot{\vec{r}}'.$$

(1 points)

One may note that $\vec{F}_{\text{Cen}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}')$ is nothing but the centrifugal force, and $\vec{F}_{\text{Cor}} = -2m\vec{\omega} \times \dot{\vec{r}}'$ the Coriolis force acting on the particle. In other words, these are the fictitious forces that appear to affect the free particle, from the point of view of the non-inertial coordinate system (x', y', z').

3 A compensatory cautionary tale (just to make sure that you do not sue me for promoting debaucheries)





Alice and Bob applied their happiness optimization very thoroughly (see problem 5.2). A bit too thoroughly perhaps, since they passed out and entered into a delirious nightmare[2]. In this dream they are trapped inside a two-dimensional funnel-like surface. If they slide up to the upper edge of the funnel, then they will leave their nightmare, while at the bottom of the funnel their student colleague 'Luci'[3] is waiting eagerly to give them a very warm welcome.

We regard Alice and Bob as point-masses m that slide without friction along a surface that is obtained by rotating the curve $z = -\frac{\alpha}{2r^2}$ for $r_0 \leq r \leq 3r_0$ around the z-axis, where $\alpha > 0$ and $r_0 > 0$. The mass m is affected by the constant gravitational acceleration g downwards along the z-axis. We want to figure out if Alice and Bob at some point will reach r_0 (and meet their friend Luci), or whether they will reach $3r_0$ (in which case they get free from their nightmare), or whether they will slide around in the region $r_0 < r < 3r_0$ for all eternity.

a) *Determine the Lagrangian of the mass m in terms of the coordinates r and θ.* **(2 points)**

[2] Beware children! This is what happens if you indulge in excessive drinking!

[3] Alice and Bob do not really know him, but everyone calls him 'Luci', and he is an exchange student from somewhere with a very hot climate. He studies the collaborative Physics-Dictatorship master-program in Cologne (part of the collaborative research center CRC 666), where he apparently is the shining light.

b) *Use the conservation of energy in order to find an first-order differential equation in r (i.e. the equation can contain r and $\dot{r}$, but not $\ddot{r}$ or higher derivatives).* **(2 points)**

Hint: One can do this via the Euler-Lagrange equation for r, but it is easier to directly use the conservation of energy. You need to use the cyclicity in order to explicitly eliminate $\dot{\theta}$ from the equation. **(2 points)**

c) We assume that the mass m starts at $r(0) = 2r_0$, at some angle $\theta(0) = \theta_0$, and that the initial angular velocity is

$$\dot{\theta}(0) = \frac{\sqrt{g\alpha}}{4r_0^2}.$$

Show that for these initial conditions, the differential equation in b) reduces to

$$\frac{m}{2}(1 + \frac{\alpha^2}{r^6})\dot{r}^2 = E. \tag{1}$$

(2 points)

d) *For the initial conditions in c), determine the fate of Alice and Bob for all possible initial values of the radial velocity $\dot{r}$.*

Hint: Equation (1) is separable, and can in principle be solved in terms of an integral, but it can be rather tricky to evaluate such integrals explicitly. However, we are not really interested in the exact solution, but rather in Alice and Bob's general destiny. One trick would be to change from r to a new coordinate, for which this question is easier to analyze. Maybe something with path-length could be useful. **(2 points)**