Classical Mechanics

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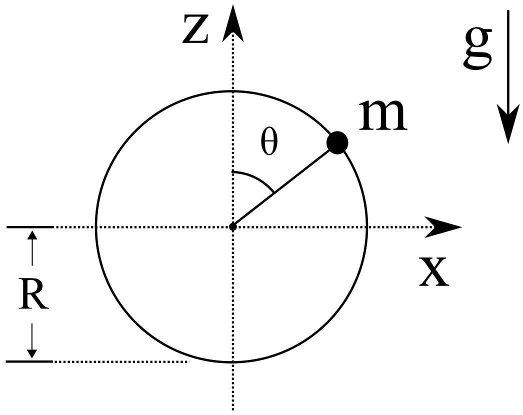
Exercise sheet 8 **Due: December, 7 at 12:00**

1 Constraint forces: The ideal case



In exercise 2 we will analyze constraint forces, but as a preparation we here consider the ideal case. We consider a bead of mass m that can slide without friction on a hoop that is oriented vertically, so that the mass m is affected by the constant gravitational acceleration g.

a) *Derive the Lagrangian in terms of the angle θ, and obtain the corresponding Euler-Lagrange equation.*

(2 points)

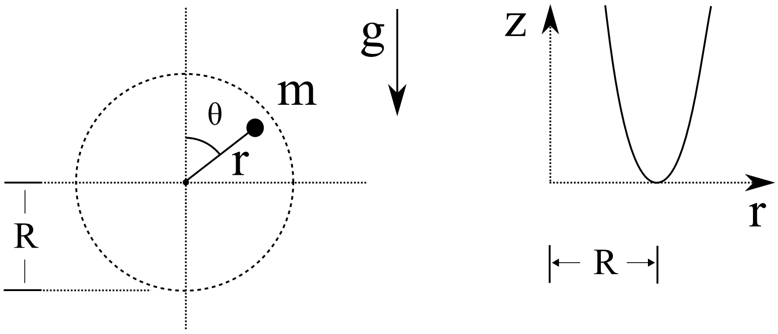
b) Suppose that the bead at a given moment of time is at angle θ and has the angular speed $\dot{\theta}$. In order to keep the bead at the constant radius R, the hoop must exert a normal force $\vec{F}_N = F_N\hat{r}$ on the bead, where $\hat{r}$ is the unit radial vector. *Show that*

$$F_N = mg\cos\theta - mR\dot{\theta}^2. \tag{1}$$

Hint: This is a bit of Newtonian mechanics again. The bead is affected by the gravitational force and the normal force. In order to stay on the hoop, the bead must experience a centripetal acceleration. This acceleration fixes the radial component of the total force acting on the bead.

(2 points)

2 Obtaining constraint forces



In the previous exercise we took for granted that the bead stays at the fixed radius R of the hoop. In other words, we assume that there are constraint forces that keep the bead precisely at the right radius. One can realize that this is an idealization. The restoring force comes from the hoop pushing the bead back, when the bead deviates slightly from the correct radius. Here we will analyze this problem by replacing the perfect constraint with a potential. We will see that one can regain the ideal constraint for very steep potentials.

Precisely as in problem 1 we consider a bead on a hoop, but we additionally allow the bead to change its radial coordinate r. Hence, we now have two coordinates θ and r. However, we also include a radial potential (in addition to the gravitational potential) $V(r) = \frac{1}{2\epsilon}(r-R)^2$, where $\epsilon > 0$. Hence, this is a quadratic potential with minimum at $r = R$, and this potential becomes more and more steep the smaller ϵ is.

The solutions θ and r to the Euler-Lagrange equations are functions of time, but also of the parameter ϵ. Intuitively is seems likely that if ϵ is very small (and thus the potential very steep), then $r(\epsilon, t) \approx R$, and we would regain the ideal case in exercise 1. In order to analyze this we are going to apply a very common type of trick, namely a perturbation expansion.

a) *Derive the Lagrangian of this system, and show that the Euler-Lagrange equations are*

$$\begin{aligned} mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = 0, \\ m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta + \frac{1}{\epsilon}(r-R) = 0. \end{aligned} \tag{2}$$

(3 points)

b) Next we expand r and θ in power-series around $\epsilon = 0$,

$$\begin{aligned} \theta(\epsilon, t) =& \theta_0(t) + \epsilon\theta_1(t) + \epsilon^2\theta_2(t) + \cdots, \\ r(\epsilon, t) =& r_0(t) + \epsilon r_1(t) + \epsilon^2 r_2(t) + \cdots. \end{aligned}$$

Insert the above expansions into the equations (2). *For each equation, collect terms of equal order in ϵ. We only need the orders $\frac{1}{\epsilon}$ and 1, so you can ignore ϵ, ϵ^2, ϵ^3, etc.*

Hint: You will get one equation from the $\frac{1}{\epsilon}$-order, and two equations from the 1-order.

(3 points)

c) Simplify the equations that you obtained in b). *How does the result relate to the Euler-Lagrange equation in problem 1 a)?* The radial potential yields the force $F(t) = -V'\big(r(t)\big)$. *For the approximation $r(t) \approx r_0(t) + \epsilon r_1(t)$, determine the force, and compare with* (1). **(2 points)**

Remark: Here we only consider the lowest orders of perturbation. For estimating how the bead deviates from the ideal evolution of the perfect constraints, one can include higher orders in the expansion (such as θ_1 and r_2). Unfortunately, the resulting expressions are not very pleasant.

3 Noether's theorem, symmetry, and change of variables

Suppose that a system has the Lagrangian

$$L(\lambda, \mu, \dot{\lambda}, \dot{\mu}) = \frac{m}{2}(\lambda^2 + \mu^2)(\dot{\lambda}^2 + \dot{\mu}^2) - \alpha \lambda^2 \mu^2, \tag{3}$$

with respect to the generalized coordinates λ and μ, and some constant α.

a) Consider the coordinate transformation from (λ, μ) to (λ', μ') where

$$\begin{aligned} \lambda' &= \lambda + sC_\lambda, \quad C_\lambda = \frac{1}{2}\frac{\lambda}{\lambda^2 + \mu^2}, \\ \mu' &= \mu + sC_\mu, \quad C_\mu = \frac{1}{2}\frac{-\mu}{\lambda^2 + \mu^2}. \end{aligned} \tag{4}$$

Show that this is a symmetry transformation of the Lagrangian L in (3) to the first order in s.

Hint: What we need to show that if $L(\lambda', \mu', \dot{\lambda}', \dot{\mu}')$ is expanded around $s = 0$, then the first order term vanishes. In other words, $L(\lambda', \mu', \dot{\lambda}', \dot{\mu}') = L(\lambda, \mu, \dot{\lambda}, \dot{\mu}) + O(s^2)$. Another way to phrase the same thing is to say that $\frac{d}{ds} L(\lambda', \mu', \dot{\lambda}', \dot{\mu}')\big|_{s=0} = 0$.

If you would get lost in the expansions and not manage to solve this problem, then you can still do problems b) and c)! **(3 points)**

b) *Compute the conserved quantity that corresponds to the transformation in (4)!* **(2 points)**

c) Consider the new coordinate system

$$\begin{aligned} x &= \lambda^2 - \mu^2, \\ y &= 2\lambda\mu. \end{aligned}$$

Express the Lagrangian (3) in terms of the new coordinates x and y. This Lagrangian has a cyclic coordinate. How does that cyclic coordinate relate to the conserved quantity in b)? **(3 points)**