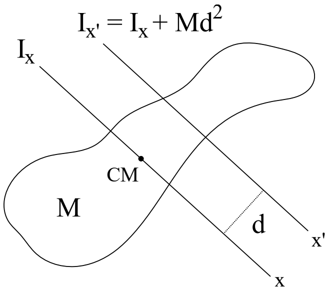
Classical Mechanics

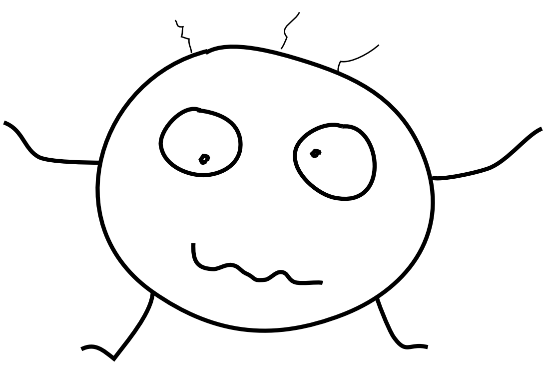
David Gross, Johan Åberg, Markus Heinrich

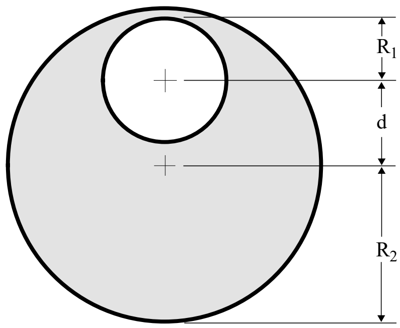
Exercise sheet 10 **Due: December, 20 at 12:00**

1 Inertia tensor of object Å

When calculating moments of inertia one can sometimes save a lot of work by using some simple tricks. One of these tricks goes under the name "parallel axis theorem" (or "Steiner's theorem"), which says the following: Suppose that a rigid body of mass M has the inertia I_x with respect to an axis x through the center of mass. If an axis x' is parallel to x with a distance d, then the moment of inertia with respect to x' is $I_{x'} = I_x + Md^2$. Another useful observation is that if a body C can be obtained by adding a body B to a body A, then $I_C = I_A + I_B$. Similarly, if C can be obtained by subtracting B from A, then $I_C = I_A - I_B$.







To the left is displayed a realistic representation of object Å. This body has total mass M, and is accurately approximated by a sphere of radius R_2 with uniform density, wherein there is a spherically shaped cavity (up towards the head) containing absolute vacuum. This spherical cavity has radius R_1, and its center is displaced by a distance d from the center of the large sphere. We have $R_2 \geq R_1 + d$, i.e., the cavity lies completely within the larger sphere.

a) *How are the principal axes of object Å oriented?* **(1 points)**

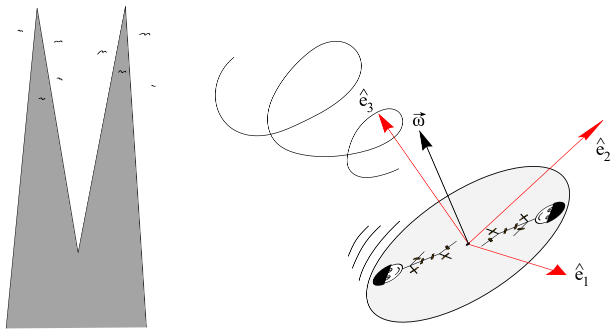
b) *Determine the inertia tensor, with respect to a coordinate system with origin at the center of the big sphere, and oriented along the principal axes.*

(5 points)

Comment: This exercise illustrates inertia tensors and some techniques to calculate them.

2 Alice and Bob on new adventures

Alice and Bob have just been on a lecture on rigid bodies, where they got very intrigued by the notion that Frisbees wobble while they spin. However, they did not really understand the theory, so they want to investigate empirically how fast the Frisbee wobbles compared to the spin. Their plan is to build a giant Frisbee, strap themselves onto it, and shoot it out from the top of the two towers of the cathedral that so helpfully has been erected in the middle of their town.[1] Perhaps you could help Alice and Bob to obtain the answer by a somewhat less drastic theoretical alternative.



The Frisbee is a special case of free symmetric top. That the top is "free" means that there is no torque acting upon it, which implies that the total angular momentum vector is conserved.[2] That the the top is "symmetric" means that two of the principal moments around the center of mass are equal, i.e., $I_1 = I_2 \neq I_3$. In the following we use the notation $I = I_1 = I_2$. Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ denote the unit vectors pointing in the direction of principal axes that are fixed to the body.[3] With respect to these we can expand the angular velocity as

$$\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3. \tag{1}$$

The evolution of $\omega_1, \omega_2, \omega_3$ describe how the angular velocity evolves from the point of view of Alice and Bob.

a) *Write down the Euler equations for the free symmetric top, and determine the general solution.* You will find that (ω_1, ω_2) traces out a circle. *Determine the angular frequency Ω of this circular motion.*

(4 points)

b) *Assuming that the Frisbee is disk of uniform mass, determine how Ω depends on ω_3.* You can look up the moments of inertia of the disk instead of calculating them.

(2 points)

Comment: The main purpose of this exercise is to introduce the Euler equations.

[1] When they posted this idea on their facebook account, "The Kamikaze physicists", they got many "likes", and their student colleague "Luci" (see sheet 9) left several very encouraging comments, so they feel confident that this is a great idea.

[2] A real Frisbee is of course affected by various forces, like friction, which also could cause torques, but we ignore this here.

[3] Note that these were called $\vec{h}_1(t), \vec{h}_2(t), \vec{h}_3(t)$ in the lecture.

3 The Frisbee in the lab frame

Alice and Bob are a bit confused by the solution you obtained in the previous exercise. Was it not said in the lecture that the Frisbee is supposed to wobble at twice the speed of the spin? You probably all realize that the reason for the discrepancy is that in the previous exercise we looked at the evolution of the vector $\vec{\omega}$ with respect to Alice and Bob's non-inertial frame $\hat{e}_1, \hat{e}_2, \hat{e}_3$ that is fixed to the Frisbee, while the comparison of the wobble and the spin in the lecture referred to someone standing still (being inertial) looking at the Frisbee. We could in principal use Euler angles to translate the result from the body-fixed frame to the lab-frame. However, this can be confusing, so we will follow another route.

In the following we will show that $\hat{e}_3$ traces out a circular motion, and determine the relation between the speed of this wobble and ω_3. Finally, we consider the special case that the spin around $\hat{e}_3$ is very fast and apply it to our idealized model of the Frisbee.

a) Let $L = |\vec{L}|$ and $\hat{L} = \vec{L}/L$ (i.e., L is the magnitude of $\vec{L}$ and $\hat{L}$ is the direction). *Derive the relation*

$$\vec{\omega} = \frac{L}{I}\hat{L} - \tilde{\Omega}\hat{e}_3, \tag{2}$$

and determine the constant $\tilde{\Omega}$ in terms of some of I, I_3 and $\omega_1, \omega_2, \omega_3$.

Hint: Keep in mind that $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are principal axes of the body, and use the relation between the angular momentum $\vec{L}$ and the angular velocity $\vec{\omega}$. Combine this with (1).

(2 points)

b) *Use (2) to show that*

$$\frac{d\hat{e}_3}{dt} = \frac{L}{I}\hat{L} \times \hat{e}_3. \tag{3}$$

Hint: Recall that the instantaneous motion of any vector $\vec{v}$ that is fixed to the rotating body is given by $\frac{d\vec{v}}{dt} = \vec{\omega} \times \vec{v}$. Note that you can solve this problem even if you did not manage to determine the constant $\tilde{\Omega}$ in a)!

(2 points)

c) As mentioned above, $\vec{L}$ is constant since we are dealing with a free top (and thus both the magnitude L and the direction $\hat{L}$ are fixed). Since $\hat{L}$ is constant, it follows that (3) describes the steady circular motion of $\hat{e}_3$ around the fixed vector $\hat{L}$ with angular velocity L/I. *Find the general expression for L in terms of $\omega_1, \omega_2, \omega_3$ and I, I_3.*

(2 points)

d) Assume now that the object is spinning very fast around axis $\hat{e}_3$ compared to the other axes, i.e., assume that $\omega_3 \gg \omega_1$ and $\omega_3 \gg \omega_2$.[4] *Find an approximate expression for L/I that only involves I, I_3, ω_3.*

For the model of the Frisbee as a uniform disk (as in problem 2b) what is the ratio between the speed of the wobble, and the speed ω_3 of the spin.[5]

(2 points)

Comment: Here we follow up on the comments in the lecture about the Frisbee.

[4] A bit more precise formulation of the condition is that $I_3^2\omega_3^2 \gg I^2(\omega_1^2 + \omega_2^2)$.

[5] Here we have compared the wobble of $\hat{e}_3$ with ω_3. However, one might alternatively ask how the wobble of $\vec{\omega}$ compares with ω_3. It turns out that $\hat{e}_3$ and $\vec{\omega}$ wobble with the same frequency. To see this, first recall that ω_3 is constant. Then one can use (2) and (3) to show that $\frac{d\vec{\omega}}{dt} = \frac{L}{I}\hat{L} \times \vec{\omega}$. The rest of the argument is identical to the one above.