Classical Mechanics

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Exercise sheet 11 **Due: January, 10 at 12:00**

1 From Lagrange to Hamilton

For the following Lagrangians, derive the Hamilton functions, and the Hamilton equations.

a) From sheet 7, problem 1: Bead on a rotating hoop

$$L(\theta, \dot{\theta}) = \frac{m}{2} R^2 \dot{\theta}^2 + \frac{m}{2} R^2 \Omega^2 \sin^2 \theta + mgR \cos\theta$$

(2 points)

b) From sheet 9, problem 1: Cautionary tale

$$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2}(1 + \frac{\alpha^2}{r^6})\dot{r}^2 + \frac{m}{2} r^2 \dot{\theta}^2 + \frac{mg\alpha}{2r^2}$$

(3 points)

Comment: This entire sheet is primarily focused on the translation from the Lagrange formulation to the Hamilton formulation of mechanics, and how to obtain Hamilton's equations from the Hamilton function.

2 Rationalizing the translation

As you may have noticed in the previous exercise, it can be a bit tedious to go through the procedure to translate the Lagrangian into a Hamilton function again and again. However, in practice, certain structures often reoccur, and for these we can do the translation once and for all, so to speak. In many cases (but certainly not always[1]) the Lagrangians have the form

$$L(\vec{q}, \dot{\vec{q}}, t) = T(\vec{q}, \dot{\vec{q}}) - V(\vec{q}, t), \qquad T(\vec{q}, \dot{\vec{q}}) = \frac{1}{2} \sum_k f_k(\vec{q}) \dot{q}_k^2, \tag{1}$$

where $f_k(\vec{q})$ are some functions of the generalized coordinates $\vec{q} = (q_1, \ldots, q_N)$, and where $V(\vec{q}, t)$ is some possibly time-dependent potential.

Show that the Hamilton function can be written

$$H(\vec{q}, \vec{p}, t) = \frac{1}{2} \sum_k \frac{p_k^2}{f_k(\vec{q})} + V(\vec{q}, t),$$

where $\vec{p} = (p_1, \ldots, p_N)$ are the conjugate momenta with respect to $\vec{q} = (q_1, \ldots, q_N)$. **(3 points)**

Comment: In this exercise we go a bit beyond specific examples, and identify a general structure in the transformation. To do such things can often be useful, not only in the sense of saving work, but also because it may help one to get a clearer picture of what is happening.

[1] The Lagrangian in problem 4 is for example not on the form (1).

3 Particle in a time-dependent potential

A particle of mass m is restricted to move along a straight line. We take the coordinate q as the position of the particle along the line. The particle is affected by the time-dependent potential $V(q,t) = C\cos(\omega t)q$ for some constants C and ω.

a) *Write down the Lagrangian and the Hamilton function of the particle.* **(2 points)**

b) *Derive the Hamilton equations and solve them for the initial condition $q(0) = 0$ and $p(0) = p_0$.* **(2 points)**

Comment: This is to give an explicit example of an application of the machinery to a time-dependent potential. In addition, this happens to be a case where one actually can find the analytical solutions to Hamilton's equations of motion.

4 Particle in an electromagnetic field

For a particle with mass m, charge e, and position $\vec{r}$, which moves in an electromagnetic field, the Lagrangian can be written

$$L(\vec{r}, \dot{\vec{r}}) = \frac{m}{2}\dot{\vec{r}}^2 - e\phi(\vec{r},t) + e\vec{A}(\vec{r},t)\cdot\dot{\vec{r}}, \tag{2}$$

where $\phi(\vec{r},t)$ is a function (the scalar potential), and a vector-valued function $\vec{A}(\vec{r},t)$ (the vector-potential)[2].

Please do not panic if you are not familiar with electromagnetism and vector-potentials, think of (2) as simply being yet another (maybe strange looking) Lagrangian of a particle.

a) Introduce the Cartesian coordinates $\vec{r} = (r_1, r_2, r_3)$ and $\vec{A} = (A_1, A_2, A_3)$. *Show that the Euler-Lagrange equations obtained from the Lagrangian (2) can be written*[3]

$$m\ddot{r}_j + e\left(\frac{\partial\phi}{\partial r_j} + \frac{\partial A_j}{\partial t}\right) - e\sum_{k=1}^{3}\dot{r}_k\left(\frac{\partial A_k}{\partial r_j} - \frac{\partial A_j}{\partial r_k}\right) = 0, \quad j = 1,2,3. \tag{3}$$

Hint: It can be useful to first rewrite (2) in terms of the Cartesian components $L(\vec{r},\dot{\vec{r}}) = \frac{m}{2}\sum_{k=1}^{3}\dot{r}_k^2 - e\phi(\vec{r},t) + e\sum_{k=1}^{3}A_k(\vec{r},t)\dot{r}_k$.

(2 points)

b) Let $\chi(\vec{r},t)$ be a real-valued function (a scalar function), and suppose that we change ϕ and $\vec{A}$ into the new functions ϕ' and $\vec{A}'$ by

$$\phi' = \phi - \frac{\partial\chi}{\partial t}, \quad \vec{A}' = \vec{A} + \nabla\chi.$$

Let L' be the new Lagrangian that is obtained if we substitute ϕ and $\vec{A}$ in (2) by ϕ' and $\vec{A}'$. *Show that L and L' only differ by a total time-derivative of some function of $\vec{r}$ and t.* **(2 points)**

Comment: This type of mapping of the potentials is called a gauge-transformation. Recall that the addition of a total time-derivative to the Lagrangian does not change the Euler-Lagrange equations (and thus does not change the evolution of the system).

[2] The electric field can be obtained as $\vec{E}(\vec{r},t) = -\nabla\phi - \frac{\partial}{\partial t}\vec{A}$ and the magnetic field as $\vec{B}(\vec{r},t) = \nabla\times\vec{A}$.
[3] The more standard way of writing (3) is $m\ddot{\vec{r}} = e\vec{E} + e\dot{\vec{r}}\times\vec{B}$.

c) *Derive the Hamilton function.* **(2 points)**

d) *Derive the Hamilton equations.* **(2 points)**

Comment: This is a non-trival example of the application of the Lagrange and Hamilton formalism, and also gives a first glimpse of the notion of gauge invariance, which you will encounter in other courses.