Classical Mechanics

David Gross, Johan Åberg, Markus Heinrich

Exercise sheet 13 **Due: January, 24 at 12:00**

1 Solving the equations of motion via canonical transformations

In this exercise we shall use canonical transformations in order to solve the equations of motion.

a) Consider the transformations from (q, p) to (Q, P) defined by

$$Q = \alpha p q^{\gamma}, \quad P = \beta q^{\delta}, \tag{1}$$

where α, β, γ and δ are constants. *What are the conditions on α, β, γ and δ for (1) to be a canonical transformation?*

Hint: Recall the characterization of canonical transformations in terms of Poisson brackets, and the definition of the Poisson bracket.

(3 points)

b) *Find a canonical transformation from (q, p) to (Q, P) that transforms*

$$H(q, p) = \frac{1}{2} p^2 q^4 + \frac{1}{2q^2} \tag{2}$$

into the Hamilton function of the Harmonic oscillator

$$H(Q, P) = \frac{1}{2} P^2 + \frac{1}{2} Q^2.$$

Hint: There was a reason for why we bothered to do exercise a). **(3 points)**

c) Use the result in b) in order to derive the solutions to Hamilton's equations corresponding to (2). **(3 points)**

Comment: In the lecture you have discussed canonical transformations. Here we see how one can use them in order to solve the equations of motions of a system.

2 Generating functions for canonical transformations

Consider a function $F_1(q, Q)$ of the old coordinates q and the new coordinates Q. This function does implicitly define a canonical transformation between (q, p) and (Q, P) via the two equations

$$p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}. \tag{3}$$

The function F_1 is referred to as the generating function of the transformation.

a) Consider the function

$$F_1(q, Q) = \frac{m\omega}{2} q^2 \frac{1}{\tan Q},$$

where m and ω are some constants. *Use the relations (3) in order to express q and p as functions of Q and P.*

Hint: The relations (3) gives p and P as functions of q and Q, and you have to transform these so that you obtain q and p as functions of Q and P. Do not worry about whether the square roots are well defined, or about the sign of the roots. **(3 points)**

b) Consider the Hamilton function for the harmonic oscillator

$$H(q,p) = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2.$$

Express H in terms of the new variables Q and P. What is the solution of the corresponding equations of motion? **(2 points)**

Remark: One could transform the solutions back to the original (q,p) and thus obtain the solutions to the equations of motion of the harmonic oscillator.

Comment: Generators provides a convenient method to construct canonical transformations.

3 Generating functions for canonical transformations again

In the previous exercise we considered generating functions $F_1(q,Q)$. However, one can also use generating functions $F_2(q,P)$, $F_3(p,Q)$, or $F_4(p,P)$. As an example we are here going to consider generating functions $F_3(p,Q)$. Such functions define canonical transformations between (q,p) and (Q,P) if

$$q = -\frac{\partial F_3}{\partial p}, \quad P = -\frac{\partial F_3}{\partial Q}. \tag{4}$$

Note the difference in signs compared to (3)!

a) Consider the function

$$F_3(p,Q) = -(e^Q - 1)^2 \tan p.$$

By using (4), determine Q and P as functions of q and p.

Hint: As in the previous exercise, do not worry about square roots (or logarithms) being well defined, or which branches to take. **(3 points)**

b) *Confirm, by using Poisson brackets, that the functions $Q(q,p)$ and $P(q,p)$ obtained in a) define a canonical transformation from (q,p) to (Q,P).* **(3 points)**

Comment: The purpose of this exercise is to complement the lecture, and the previous exercise, which discussed generators on the form $F(q,Q)$. As seen here, one can also use other generators.