Classical Mechanics

David Gross, Johan Åberg, Markus Heinrich

Exercise sheet 2 **Due: October, 26 at 12:00**

1 Rocket equation

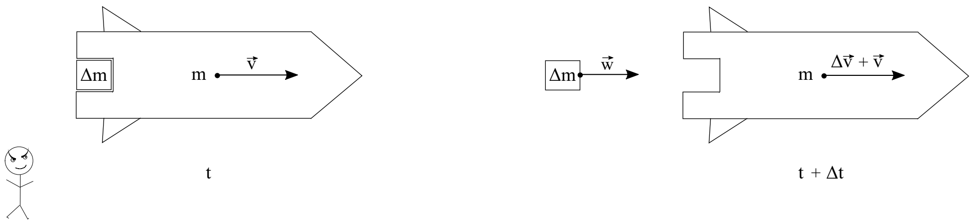

Figure 1: At time t a rocket of weight m together with a bunch of fuel Δm moves at velocity $\vec{v}$ relative to an inertial observer. At time $t+\Delta t$, the fuel Δm has been ejected, and moves at speed $\vec{w}$, relative to the same inertial observer. At the same time the rocket moves at speed $\vec{v}+\Delta\vec{v}$. To obtain the rocket equation one can consider the change of momentum of this system for small Δt.

A rocket is propelled by the reaction forces from the burning fuel. As the fuel is burnt, the rocket looses mass, which of course affects the acceleration. The rocket equation describes the evolution of the velocity of the rocket.

$$\vec{F} = m\frac{d\vec{v}}{dt} - \vec{u}\frac{dm}{dt} \tag{1}$$

Here $\vec{v}$ is the velocity of the rocket relative to an inertial observer. $\vec{u}$ is the velocity of the expelled gas *relative to the rocket*. m is the mass of the rocket (including the fuel that it carries). $\vec{F}$ is the external force (e.g. gravity) affecting the rocket. For the derivations it can be useful to also introduce the velocity $\vec{w}$ of the expelled gas relative to the inertial observer.

a) Initially, the weight of the rocket is $m+\Delta m$ and its velocity is $\vec{v}$ with respect to an inertial observer. After a time-interval Δt, the rocket has mass m and velocity $\vec{v}+\Delta\vec{v}$, while the ejected fuel-mass Δm has velocity $\vec{w}$ with respect to the inertial observer. *What is the change of momentum, of the combined system of the rocket and the ejected mass, during the time-interval Δt?*

(3 points)

b) *Use the result in (a) to derive (1).* **(3 points)**

Hint: Consider the limit $\Delta t \to 0$. What is the relation between force and the time-derivative of the momentum? You should obtain an equation similar (1), but which requires some further manipulations. It is a good idea to figure out the relations between $\vec{w}$, $\vec{u}$, and $\vec{v}$. In the limit of small Δt, what is the relation between $\frac{\Delta m}{\Delta t}$ and $\frac{dm}{dt}$?

c) Assume that the rocket is in free space, and hence $\vec{F}=0$, and that the exhaust velocity $\vec{u}$ is constant. Assume moreover that the rocket initially has zero velocity relative to the inertial observer. *Express the final velocity of the rocket in terms of the exhaust velocity $\vec{u}$, as well as the initial mass m_i and final mass m_f of the rocket. Does the final velocity depend on the time that it takes to reach the final mass?* **(3 points)**

d) Now assume that we send the rocket upwards in a constant gravitational field (directed downwards) and with a constant exhaust velocity (directed downwards), and where the initial velocity of the rocket is zero. *Again find an expression of the final velocity. Does the final speed depend* *on the time that it takes to reach the final mass? With your answer in mind, can you explain why there is such a spectacular burning of fuel at rocket launches (apart from dictators thinking that it looks cool)?*

(3 points)

Comment: The purpose of this exercise is to explore the consequences of conservation of momentum.

2 Application of Kepler's laws

Kepler's first law states that the planets move along ellipses around the sun[1]. One way to describe the orbit is by using polar coordinates, where the radius r depends on the angle θ as

$$r(\theta) = \frac{p}{1+\epsilon\cos\theta}, \qquad (2)$$

where ϵ is the eccentricity of the ellipse, and p gives the size of the orbit. Kepler's second law says that the line joining the planet and the sun sweeps out equal areas during equal intervals of time. Kepler's third law states the length a of the semimajor axis of the ellipse relates to the period T of the orbit as

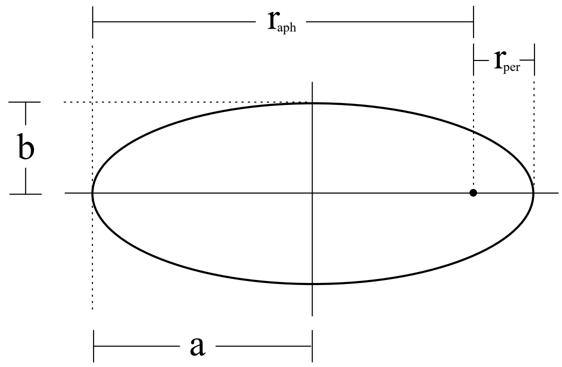

Figure 2: The semimajor axis a and the semiminor axis b of the ellipse. The shortest distance r_{per} to the sun (perihelion), and the longest distance r_{aph} (aphelion).

$$\frac{a^3}{T^2} = \frac{G(M+m)}{(2\pi)^2},$$

where M and m are the mass of the sun and the planet, respectively. Note that $G \approx 6.67 * 10^{-11} Nm^2/kg^2$.

a) Before landing, Apollo 11 was put in orbit around the moon. The mass of of Apollo 11 was 9979 kg and the period of the orbit was 119 min. The maximum and minimum distances from the center of the moon were 1861 km and 1838 km. *Use these data to estimate the mass of the moon.* **(2 points)**

b) Halley's comet moves in an elliptic orbit around the sun, with a period of 76 years. The eccentricity is $\epsilon = 0.97$. The mass of the sun is about $M = 2.0 * 10^{30} kg$. *Use these data to determine the distance from the sun at perihelion (when the comet is the closest to the sun) and aphelion (when it is the most far away).*

Hint: We can ignore the mass of the comet compared to the mass of the sun. **(3 points)**

c) *What is the ratio of the speed of Halley's comet when it is in perihelion compared with when it is in aphelion?* It is useful to keep in mind that $r^2 \frac{d\theta}{dt} =$ constant (which comes from the conservation of angular momentum).

Hint: What is the radial speed at perihelion and aphelion? How does the tangential speed relate to the angular speed and r?

(3 points)

Comment: The purpose of this exercise is to demonstrate some uses of Kepler's laws.

[1] Kepler's laws is not only applicable to things orbiting around the sun, but can also be applied also to other constellations. Strictly speaking, Eq. (2) describes the motion around the center of mass of the two bodies. However, in our case $M \gg m$ and thus the center of mass is approximately the position of the heavier body.