Classical Mechanics

David Gross, Johan Åberg, Markus Heinrich

Exercise sheet 4 **Due: November, 8 at 12:00**

1 The Laplace-Runge-Lenz vector

You have heard about conserved quantities such as energy, momentum, and angular momentum, but sometimes a system can also possess more 'exotic´ types of conserved quantities. An example turns out to be the Kepler problem, i.e., the motion of a particle in a potential of the form $V(\vec{r}) = -\alpha/r$, for $r = \|\vec{r}\|$, and for some constant α. Like for any central symmetric potential, the Kepler problem conserves the total energy and the angular momentum with respect to the center of the potential. However, it turns out that it also possesses an additional 'accidental' conserved quantity, namely the Laplace-Runge-Lenz vector, which is defined as

$$\vec{A} = \vec{p} \times \vec{L} - m\alpha\frac{\vec{r}}{r}.$$

Show that $\vec{A}$ is conserved, i.e., show that $\frac{d\vec{A}}{dt} = 0$.
Hint: Keep in mind that $\vec{L}$ is conserved. Make use of the general relation $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$. Show that $\vec{r} \cdot \frac{d\vec{r}}{dt} = r\frac{dr}{dt}$. **(6 points)**
Comment: In the lecture it was claimed that the Laplace-Runge-Lenz vector is conserved. Here we confirm that this indeed is true.

.

2 Rotating reference frame

Consider the following everyday situation (as shown in the figure). After having studied hard long into the night before your lecture in theoretical physics, you fall asleep on the merry-go-round in Klettenbergpark. When you wake up next morning, the turntable is still spinning with constant angular velocity ω.

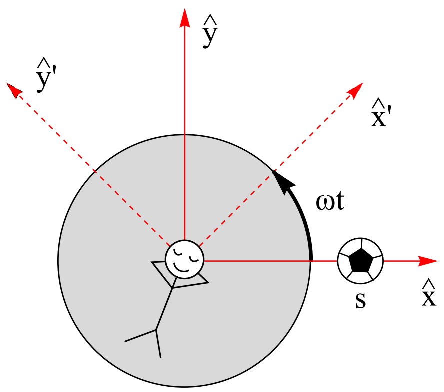

Figure 1: You are lying on a Merry-go-round that rotates with constant angular velocity ω. To the Merry-go-round you have fixed a coordinate system with unit vectors $\hat{x}', \hat{y}'$ that thus rotates with respect to a coordinate system $\hat{x}, \hat{y}$ fixed to the ground. With respect to $\hat{x}, \hat{y}$ a ball has the coordinates $r_0 = (x_0, y_0)^T = (s, 0)^T$.

a) Fixed to the merry-go-round, your coordinate system $\hat{x}', \hat{y}'$ rotates with constant angular velocity ω with respect to $\hat{x}, \hat{y}$ (see the figure). We will denote the coordinates in the two coordinate systems by $\boldsymbol{r} = (x, y)^T$ and $\boldsymbol{r}' = (x', y')^T$, respectively, such that[1]

$$\vec{r} = x\hat{x} + y\hat{y} = x'\hat{x}' + y'\hat{y}'.$$

Derive the transformation between the vectors $\hat{x}, \hat{y}$ and $\hat{x}', \hat{y}'$. What is the transformation law between the corresponding coordinates? **(4 points)**

b) A bit away from the merry-go-round lies a ball that is at rest with respect to the inertial coordinate system (with unit vectors $\hat{x}, \hat{y}$). The coordinates of the ball with respect to this coordinate system are $\boldsymbol{r}_0 = (x_0, y_0)^T = (s, 0)^T$ for $s > 0$. *What are the coordinates $\boldsymbol{r}'_0 = (x'_0, y'_0)^T$ of the ball with respect to your rotating coordinate system?*

Suddenly a child comes running and kicks the ball such that it moves (without friction) with constant speed v in the $\hat{y}$-direction (i.e., the velocity is $v\hat{y}$). *Determine the coordinates $\boldsymbol{r}'_1 = (x'_1, y'_1)^T$ of the moving ball with respect to your rotating coordinate system.* **(3 points)**

c) *Draw the curves that the coordinates $\boldsymbol{r}'_0$ and $\boldsymbol{r}'_1$ trace out in the plane.* A qualitative sketch is enough, but indicate the direction of the motion with some arrows. **(2 points)**

d) Suppose that when you wake up you have forgotten that you are lying on a merry-go-round, and believe that you are in an inertial coordinate system. *What forces $\boldsymbol{F}'_0$ and $\boldsymbol{F}'_0$ would you need to assume, such that Newton's law $\boldsymbol{F}'_j = m\frac{d^2}{dt^2}\boldsymbol{r}'_j$ would be satisfied for $j = 0, 1$?* **(5 points)**

Comment: This exercise highlights the difference between inertial and non-inertial coordinate systems.

[1]Here, ^{T} denotes transposition.