Classical Mechanics

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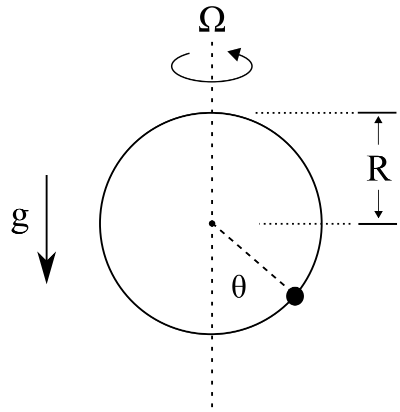
Exercise sheet 7 **Due: November, 29 at 12:00**

1 Bead on a rotating hoop



A bead of mass m slides without friction on a hoop of radius R that rotates around its vertical axis with a constant angular speed Ω.

a) *Derive the Lagrangian expressed in terms of the angle θ that the bead makes with the vertical downwards direction.* **(2 points)**

b) The Lagrangian that you obtain is equivalent to that of a particle of mass M that moves along the real line, and being affected by a (periodic) potential $V(\theta)$. *Determine M and $V(\theta)$.*

(1 points)

c) *Find the Euler-Lagrange equation of the bead.* **(2 points)**

d) The equilibrium configurations of the bead depend on the angular speed Ω. *As a function of Ω, determine the equilibrium configurations, and show whether they are stable or unstable.*

Hint: Which points are stable or unstable depends on Ω.

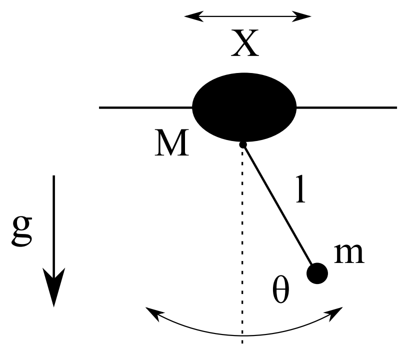
Remark: Recall that if a particle moves in a potential V, then the equilibrium positions of the particle are those where the derivative (or more generally the gradient) of the potential V is zero. This means that if we put the particle at rest at such a point, it will remain there. Recall also that the equilibrium is stable if the potential has a strict local minimum at the equilibrium point, which means that for all sufficiently small deviations, the particle will remain in a bound motion around the equilibrium. However, for an unstable equilibrium, the particle can move away at the slightest perturbation.[1]

(4 points)

Comment: The primary purpose of this exercise is to train the Lagrangian method, but also to introduce the concepts of stable and unstable equilibrium solutions, and how these can depend on external parameters.

[1] Strictly speaking, we also have the case of a neutral equilibrium, if the potential is flat.

2 Pendulum on a sliding support



An object of mass M can slide without friction along a horizontal rail. From the object hangs a massless rod of length l with a mass m attached to the end. The rod can swing (without friction) in the plane along the rail and the vertical direction.

a) *Derive the Lagrangian in terms of the position X of the mass M, and the angle θ.* **(3 points)**

b) *Derive the Euler-Lagrange equations.* **(3 points)**

c) *Find the approximate equation in the the small angle approximation.* This means that we assume that θ is so small that we can make the approximation $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$, and that we in the resulting equations only keep terms up to first order in θ, $\dot{\theta}$, and $\ddot{\theta}$ (e.g. θ^2 and $\theta\dot{\theta}$ are second order terms, while $\theta\dot{\theta}^2$ is a third order term). **(2 points)**

d) *Find the complete solution to the approximate equations of motion in c). What are the normal modes, and the corresponding frequencies?* **(3 points)**

Comment: Again, the main purpose is the Lagrangian method, but also to reconnect to the notion of normal modes via the approximate harmonic motion near an equilibrium.