Classical Mechanics

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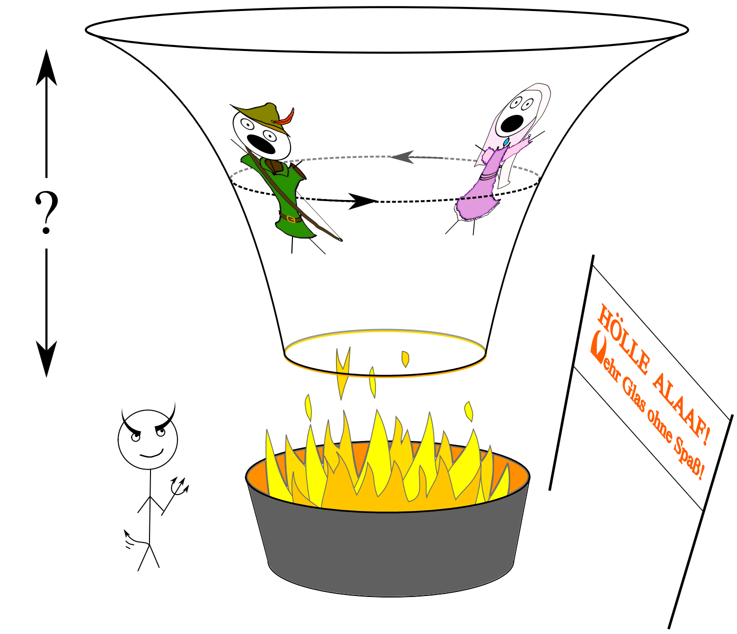
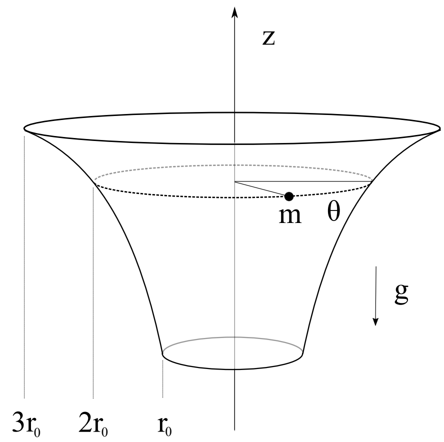
Exercise sheet 9 **Due: December, 13 at 12:00**

1 A compensatory cautionary tale (just to make sure that you do not sue me for promoting debaucheries)





Alice and Bob applied their happiness optimization very thoroughly (see problem 6.2). A bit too thoroughly perhaps, since they passed out and entered into a delirious nightmare[1]. In this dream they are trapped inside a two-dimensional funnel-like surface. If they slide up to the upper edge of the funnel, then they will leave their nightmare, while at the bottom of the funnel their student colleague 'Luci'[2] is waiting eagerly to give them a very warm welcome.

We regard Alice and Bob as point-masses m that slide without friction along a surface that is obtained by rotating the curve $z = -\frac{\alpha}{2r^2}$ for $r_0 \leq r \leq 3r_0$ around the z-axis, where $\alpha > 0$ and $r_0 > 0$. The mass m is affected by the constant gravitational acceleration g downwards along the z-axis. We want to figure out if Alice and Bob at some point will reach r_0 (and meet their friend Luci), or whether they will reach $3r_0$ (in which case they get free from their nightmare), or whether they will slide around in the region $r_0 < r < 3r_0$ for all eternity.

a) *Determine the Lagrangian of the mass m in terms of the coordinates r and θ.* **(2 points)**

b) *Use the conservation of energy in order to find an first-order differential equation in r (i.e. the equation can contain r and $\dot{r}$, but not $\ddot{r}$ or higher derivatives).* **(2 points)**

Hint: One can do this via the Euler-Lagrange equation for r, but it is easier to directly use the conservation of energy. You need to use the cyclicity in order to explicitly eliminate $\dot{\theta}$ from the equation. **(2 points)**

[1] Beware children! This is what happens if you indulge in excessive drinking!

[2] Alice and Bob do not really know him, but everyone calls him 'Luci', and he is an exchange student from somewhere with a very hot climate. He studies the collaborative Physics-Dictatorship master-program in Cologne (part of the collaborative research center CRC 666), where he apparently is the shining light.

c) We assume that the mass m starts at $r(0) = 2r_0$, at some angle $\theta(0) = \theta_0$, and that the initial angular velocity is

$$\dot{\theta}(0) = \frac{\sqrt{g\alpha}}{4r_0^2}.$$

Show that for these initial conditions, the differential equation in b) reduces to

$$\frac{m}{2}(1 + \frac{\alpha^2}{r^6})\dot{r}^2 = E. \tag{1}$$

(2 points)

d) *For the initial conditions in c), determine the fate of Alice and Bob for all possible initial values of the radial velocity $\dot{r}$.*

Hint: Equation (1) is separable, and can in principle be solved in terms of an integral, but it can be rather tricky to evaluate such integrals explicitly. However, we are not really interested in the exact solution, but rather in Alice and Bob's general destiny. One trick would be to change from r to a new coordinate, for which this question is easier to analyze. Maybe something with path-length could be useful. **(2 points)**

Comment: In this exercise we combine various components that we have seen before, such as Euler-Lagrange equations, energy conservation, and conservation of angular momentum.

2 Noether's theorem, symmetry, and change of variables

Suppose that a system has the Lagrangian

$$L(\lambda, \mu, \dot{\lambda}, \dot{\mu}) = \frac{m}{2}(\lambda^2 + \mu^2)(\dot{\lambda}^2 + \dot{\mu}^2) - \alpha\lambda^2\mu^2, \tag{2}$$

with respect to the generalized coordinates λ and μ, and some constant α.

a) Consider the coordinate transformation from (λ, μ) to (λ', μ') where

$$\begin{aligned} \lambda' &= \lambda + sC_\lambda, \quad C_\lambda = \frac{1}{2}\frac{\lambda}{\lambda^2 + \mu^2}, \\ \mu' &= \mu + sC_\mu, \quad C_\mu = \frac{1}{2}\frac{-\mu}{\lambda^2 + \mu^2}. \end{aligned} \tag{3}$$

Show that this is a symmetry transformation of the Lagrangian L in (2) up to the first order in s.

Hint: A family of transformations $\vec{\Phi}^{(s)} : \mathbb{R}^n \to \mathbb{R}^n$ with $\vec{\Phi}^{(0)}(\vec{q}) = \vec{q}$ is a symmetry transformation of a Lagrangian $L(\vec{q}, \dot{\vec{q}})$ up to the first order in s if $L\Big(\vec{\Phi}^{(s)}(\vec{q}(t)), \frac{d}{dt}\vec{\Phi}^{(s)}(\vec{q}(t))\Big) = L(\vec{q}(t), \dot{\vec{q}}(t)) + O(s^2)$, or equivalently if $\frac{d}{ds}\Big|_{s=0} L\Big(\vec{\Phi}^{(s)}(\vec{q}(t)), \frac{d}{dt}\vec{\Phi}^{(s)}(\vec{q}(t))\Big) = 0.$[3] As you may recall from the lecture, this is an assumption of Noether's theorem.

Remark: If you would get lost in the expansions and not manage to solve this problem, then you can still do problems b) and c)! **(4 points)**

b) *Compute the conserved quantity that corresponds to the transformation in (3).* **(3 points)**

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[3] This can be compared with exact symmetry transformations, where $L\Big(\vec{\Phi}^{(s)}(\vec{q}(t)), \frac{d}{dt}\vec{\Phi}^{(s)}(\vec{q}(t))\Big) = L(\vec{q}(t), \dot{\vec{q}}(t))$ for all s. One can realize that an exact symmetry is also a symmetry up to the first order in s.

c) Consider the new coordinate system

$$x = \lambda^2 - \mu^2,$$
$$y = 2\lambda\mu.$$

Express the Lagrangian (2) in terms of the new coordinates x and y. This Lagrangian has a cyclic coordinate. How does that cyclic coordinate relate to the conserved quantity in b)? **(3 points)**

Comment: This exercise illustrates Noether's theorem for something else than translations or rotations.