**14756: Quantum Error Correction** **WS 2018**

Solution 2

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## Excercise 1

**Solution to 1.1** Density matrix is a positive semi-definite Hermitian matrix. As it is Hermitian, it is diagonalizable as $\rho = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle \langle\psi_i|$ where $\{|\psi_i\rangle\}$ are orthgonal state vectors. As $\rho$ is a positive semi-definite, $\lambda_i \geq 0$ for all $i$ and $\mathrm{Tr}[\rho] = 1 = \sum_i \lambda_i$. Then $\mathrm{Tr}[\rho^2] = \sum_i \lambda_i^2 \leq \sum_i \lambda_i = 1$ as $0 \leq \lambda_i \leq 1$. When $\sum_i \lambda_i^2 = 1$, $0 = \sum_i \lambda_i - \lambda_i^2 = \sum_i \lambda_i(1-\lambda_i)$. As each term in the summation is non-negative, $\lambda_i - \lambda_i^2$ should be 0. Thus $\lambda_i = 0$ or 1. Again, as $\mathrm{Tr}[\rho] = 1$, there is one $i$ that $\lambda_i = 1$ and others should be 0. This means that $\rho$ is a rank-one matrix (pure state).

**Solution to 1.2**
1. As $\rho$ is Hermitian and in a 2 dimensional linear space, there is $U \in \mathrm{SU}(2)$ that diagonalize $\rho$ in a computational basis ($\sigma_z$ basis), i.e. $U^\dagger \rho U = a|0\rangle\langle 0| + (1-a)|1\rangle\langle 1| = (\mathbb{1} + r\sigma_z)/2$ where $r = 2a - 1$. From positive semi-definite property of $\rho$, $0 \leq a \leq 1$. Then $\rho = (\mathbb{1} + rU\sigma_z U^\dagger)/2$. As there is a unit vector $\hat{r}$ such that $U\sigma_z U^\dagger = \hat{r}\cdot\vec{\sigma}$, $\rho = (\mathbb{1} + \vec{r}\cdot\vec{\sigma})/2$ where $\vec{r} = r\hat{r}$. Moreover, $\|\vec{r}\| = |r| = |2a-1| \leq 1$.

2. From the above, $\vec{r} = \mathrm{Tr}[\rho\vec{\sigma}]$. As $\langle +|Z|+\rangle = \langle -|Z|-\rangle = \langle +|Y|+\rangle = \langle -|Y|-\rangle = 0$, only $x$ component exists. And $\mathrm{Tr}[\rho X] = \alpha - (1-\alpha) = 2\alpha - 1$. Thus the Bloch vector is $(2\alpha - 1, 0, 0)$ that lies on a line segment $(-1, 0, 0)$ to $(1, 0, 0)$.

3. Direct computation yields $\rho^2 = (\mathbb{1} + 2\sum_i r_i\sigma_i + \sum_{i,j} r_i\sigma_i r_j\sigma_j)/4$. As $\mathrm{Tr}[\sigma_{x,y,z}] = 0$, we only need to know $\sum_{i,j} r_i\sigma_i r_j\sigma_j$ to obtain $\mathrm{Tr}[\rho^2]$. As $\sigma_i\sigma_j = \delta_{i,j}\mathbb{1} + i\epsilon_{ijk}\sigma_k$ where $\epsilon_{ijk}$ is the Levi-Civita symbol, $\sum_{i,j} r_i r_j \sigma_i\sigma_j = \sum_i r_i^2$ so $\mathrm{Tr}[\rho^2] = (1 + \sum_i r_i^2)/2$. From the previous problem, $\rho$ is pure iff $\mathrm{Tr}[\rho^2] = 1$ and it means $\sum_i r_i^2 = \|\vec{r}\| = 1$.

## Exercise 2

**Solution to 2.1** As in Exercise 1.2, there is $U \in \mathrm{SU}(2)$ such that $U^\dagger\sigma_z U = \vec{u}\cdot\vec{\sigma}$. Thus, eigenvalues of $\vec{u}\cdot\vec{\sigma}$ are $\pm 1$ and eigenvectors are $U^\dagger|0\rangle$ and $U^\dagger|1\rangle$. Recall that the projectors onto the computational basis are $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ that are $(\mathbb{1} \pm \sigma_z)/2$. Then the projectors onto corresponding eigenspace are $U^\dagger|i\rangle\langle i|U = (\mathbb{1} \pm U^\dagger\sigma_z U)/2 = (\mathbb{1} \pm \vec{u}\cdot\vec{\sigma})/2$.

**Solution to 2.2** The probability is given by $p(+) = \langle 0|P_+|0\rangle = (1 + u_z)/2$ where $\vec{u} = (u_x, u_y, u_z)$. The post-measurement state is given by $P_+|0\rangle/\sqrt{p(+)} = U^\dagger|0\rangle$. When we write $\vec{u}$ in the spherical coordinate as $\vec{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, this is given as $U^\dagger|0\rangle = (\cos(\theta/2), e^{i\phi}\sin(\theta/2))$.

## Exercise 3

A positive semi-definite Hermitian matrix $A$ can be decomposed as $A = \sum_k \lambda_k v_k v_k^\dagger$ where $\lambda_k \geq 0$ are eigenvalues and $v_k$ are corresponding orthogonal eigenvectors. Then $T(A) = \sum_k \lambda_k v_k^* v_k^T$. Thus eigenvalues are still $\lambda_k$ (with a corresponding eigenvector $v_k^*$) so the map $T$ is positive. However, the map is not completely positive. For a quantum state $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, the density matrix is $\rho = |\psi\rangle\langle\psi| = (|00\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)/2$. Applying transpose only to the second qubit, $(\mathrm{id} \otimes T)(\rho) = (|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|)/2$. In a matrix form, it is written as

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}. \tag{1}$$

Evaluating the eigenvalues of this matrix gives $-1/2, 1/2$ so the state is not positive. Therefore, the transpose map is not completely positive.