14756: Quantum Error Correction		**WS 2018**
	Solution 4	
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Excercise 1

Let us first consider an arbitrary single qubit phase (Z) error. For a given single qubit phase error, we obtain following syndrome outcomes:

	S_4	S_5	S_6
Z_1	-1	1	1
Z_2	1	-1	1
Z_3	-1	-1	1
Z_4	1	1	-1
Z_5	-1	1	-1
Z_6	1	-1	-1
Z_7	-1	-1	-1

As the measurement outcomes are distinct, the syndrome measurements uniquely identify the location of the error. Thus any single qubit phase error can be corrected by applying Z operator to the location that is identified from the syndrome measurements. And syndromes are symmetric under the exchange between X and Z, we can also correct an arbitrary X error as the same way.

Note The above argument implies Knil-Laflamme condition. Let $\mathbf{E} = \{X_i, Z_i\}$ be the set of possible single qubit errors and P is the projector into the codespace. Then the Knil-Laflamme condition is given by $PE_i^\dagger E_j P = c_{i,j} P$ where $C = (c_{i,j})$ is a Hermitian matrix. As the above argument says that there exists a stabilizer operator $s \in S$ that commutes with E_j but not commutes with $E_i^\dagger$ if $i \neq j$. Then $PE_i^\dagger E_j P = PE_i^\dagger E_j s P = PE_i^\dagger s E_j P = -PsE_i^\dagger E_j P = -PE_i^\dagger E_j P$. Thus $PE_i^\dagger E_j P = 0$ and $c_{i,j}$ are nonzero only when $i = j$ so C is Hermitian.

Note 2 Recall the classical Hamming code that the syndrome measurements gives the position of the error and see the table again. Can you find how it is related to the classical Hamming code?

Exercise 2

Solution to 2.1 Any two stabilizer generators s_i and s_j share 3 common qubits as their supports. Among operators that act on the same qubits, there are two anticommuting and one commuting operators. Thus any two generators commute.

Solution to 2.1 It is easy to check that $\overline{X} = X_1X_2X_3X_4X_5$ commutes with S_1 to S_4. In addition, it is impossible to generate an operator only has X by multiplying stabilizer generators. Thus $\overline{X}$ is not in S so acts as a logical operator. Likewise $\overline{Z} = Z_1Z_2Z_3Z_4Z_5$ is also an logical operator. As $S_1\overline{X} = Y_2Y_3X_5$ is also an logical operator with weight 3, the distance of the code is at most 3. One may also obtain a logical qubit from the stabilizers. First, observe that $S_5 := S_1S_2S_3S_4 = Z_1Z_2X_3X_5$ completes the cyclic permutations of

S_1. Then any product of three stabilizer generators (S_1 to S_4) can be represented as a product of two cyclic permutations of S_1. Thus, the stabilizer group has 15 elements ($\mathbb{1}$, S_1 and its cyclic permutations, S_1S_2 and its cyclic permutation, and S_1S_3 and its cyclic permutations). To obtain a +1 eigenstate of $\overline{Z}$ within the code space, we may start with $|00000\rangle$ that is also +1 eigenstate of $\overline{Z}$ but not in the code space. Then a (unormalized) logcial qubit can be obtained as

$$\begin{aligned}
|\overline{0}\rangle &= \sum_{s\in S} s\,|000000\rangle & (1)\\
&= |00000\rangle + (S_1 + \text{cyclic perms})\,|00000\rangle + (S_1S_2 + \text{cyclic perms})\,|00000\rangle \\
&\quad + (S_1S_3 + \text{cyclic perms})\,|00000\rangle & (2)\\
&= |00000\rangle + (|10010\rangle + \text{cyclic perms}) - (|11011\rangle + \text{cyclic perms}) - (|00110\rangle + \text{cyclic perms}) & (3)
\end{aligned}$$

and the other eigenstate of $\overline{Z}$ is obtained by applying $\overline{X}$ to $|\overline{0}\rangle$.

Solution to 1.3 Let $\mathbf{E}$ be a set of any single qubit error operators. To apply the Knil-Laflamme theorem, it is sufficient to show that there is a stabilizer that anticommutes with $E_i^\dagger E_j$ for $E_i, E_j \in \mathbf{E}$ with $i \neq j$. This is because for such s, we obatin $PE_i^\dagger E_j P = PE_i^\dagger E_j sP = -PsE_i^\dagger E_j P = -PE_i^\dagger E_j P$ so $PE_i^\dagger E_j P = 0$ where P is the projector into the codespace. It is simple to check that any product of two X or Z operators and X_iZ_j operators have an anticommuting stabilizer (you may use the transnational symmetry of S).