14756: Quantum Error Correction	**WS 2018**

Solution set 5

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Excercise 1

Quantum error correcting codes can correct any errors on up to $\lfloor (d-1)/2 \rfloor$ qubits when d is the distance of the code. Thus, the minimum weight of an error configuration that makes a logical error is on $\lfloor (d+1)/2 \rfloor$ qubits. In the Toric code, the code is characterized by $[[n, k, d]] = [[2L^2, 2, L]]$ so the code distance is L. Then when the optimal decoder fails in the Toric code with an error on $\lfloor (d+1)/2 \rfloor$ qubits? First, let us consider X type of errors. When there is such an error, there exists a path P that connects the error syndromes and crosses the erroneous qubits. The optimal decoder fails when there is another path P' that also connects the error syndromes and satisfies 1) $P + P'$ is a logical error operator and 2) the weight of P' is smaller than P. Because of 2), the decoder will choose P' instead of P and this choice yields a logical error due to 1).

Then let us apply the conditions above to find such a path. The logical operator must have the weight $\geq L$, the first condition says that the weight of P' should be $\geq L - \lfloor (L+1)/2 \rfloor$. At the same time, the second condition gives the weight of P' should be $\leq \lfloor (L+1)/2 \rfloor$. When L is even, as $\lfloor (L+1)/2 \rfloor = L/2$ the weight of P' ($w(P')$) is also $L/2$. Moreover, as $P + P'$ should make a connected path, all errors should lie on a straight line. For odd $L = 2k+1$, we have $w(P) = k+1$ and $k \leq w(P') \leq k+1$. However, a logical operator must have the weight $L+$(even number) or it cannot be connected. Thus we have $w(P) = k+1$, $w(P') = k$ and also the errors should lie on a straight line. An example is given in the figure.

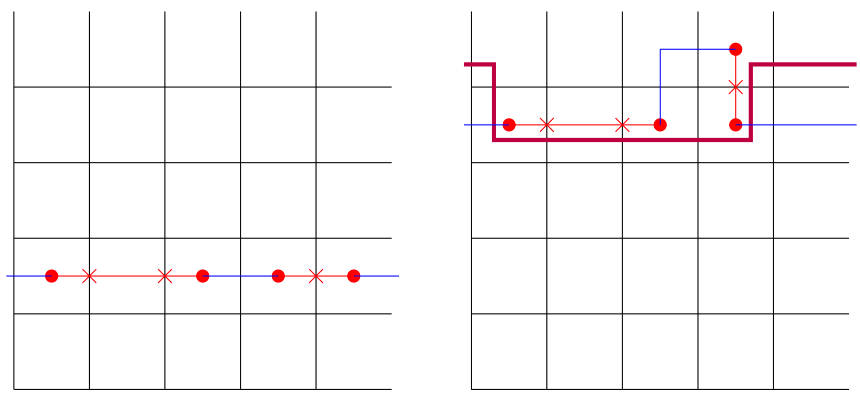


Figure 1: Left: There are $\lfloor (L+1)/2 \rfloor$ errors on a straight line. The path that connects the errors is depicted as red lines. However, there exists a shorter path (blue one) that crosses only two qubits. Choosing the blue path instead of the red path makes a logical error. Right: When errors do not lie on a straight line, a shortest logical operator that contains erroneous qubits must have the length longer than $L+2$ (e.g. purple line). As $L+2-\lfloor (L+1)/2 \rfloor$ is always larger than $\lfloor (L+1)/2 \rfloor$, there is no other path than P (red path) that the length is shorter but makes a logical error. For example, blue path has length 4 that is larger than P.

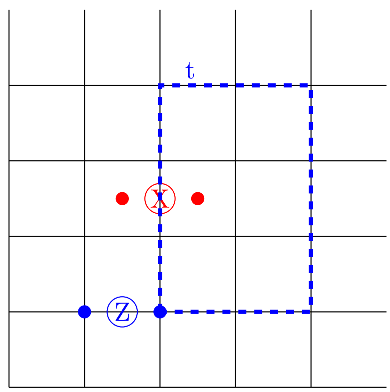
Then let us count the number of possible such error configurations. In the Toric code, we have $2L$ straight lines consist of L qubits. Among L qubits in a straight line, any errors on $\lfloor (L+1)/2 \rfloor$ qubits makes a logical error under the optimal decoder. So we have $2L \times \binom{L}{\lfloor (L+1)/2 \rfloor}$ such error configurations. The same argument also holds for Z errors.

Exercise 2

It is easy to check that for $L = 3$ Toric code, X or Z error produce a unique pair of syndromes. To be precise, let S_i be a set of X star (cross) operators and Z plaquette operators and E_i be all single X and Z operators. Then for each E_i, we have exactly two S_{i_1} and S_{i_2} that anticommutes with E_i. Moreover, this map is injective, i.e. if $\{i_1, i_2\} = \{i'_1, i'_2\}$ then $i = i'$. This implies that the code can correct an arbitrary single qubit error.

Exercise 3

In the anyon picture, each syndrome is considered as a particle. Because of the constraints in the Toric code, syndromes only can be given as a pair. To braid anyons, we need two different excitations (X and Z types). We now consider moving one of cross type (associated with X cross stablizers) particles that is excited by an Z error around one of plaquette excitations. The situation is demonstrated in the figure.



This moving corresponds to applying Z operators along the dahsed line. Because Z plaquette oprators are within the stabilizer group, this operation does not change the state if there were no X error. But when the path crosses X operator as in our case, the state obtain an overall phase -1. Consider a quantum state $|\psi\rangle$ within a code space. After we applying X and Z error in a given location, we have $X_p Z_q |\psi\rangle$ where p and q are indices for the location of errors. After applying Z operators along the dahsed line $Z(t) = \Pi_{i \in t} Z_i$, the state becomes $Z(t) X_p Z_q |\psi\rangle = -X_p Z(t) Z_q |\psi\rangle$ as $p \in t$ so $Z(t)$ and X_p anticommute. As Z_q and $Z(t)$ commute and Z_t acts trivially for a ground state, we finally obtain $-X_p Z_q |\psi\rangle$. In this discussion, the order of Z operators within $Z(t)$ does not matter. It means that the direction of braiding is not important.