14756: Quantum Error Correction **WS 2018**

Solution set 6

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Excercise 1

Solution to Exercise 1.1

We assume an error model that X and Z occur independently in each qubit with probability p so we can consider the logical failure rate only for single type of error (let's say X). Under this assumption, the logical failure probability is formally written as

$$P_{\text{fail}} = \sum_E (1-p)^{n-|E|} p^{|E|} = (1-p)^n \sum_E \big(\frac{p}{1-p}\big)^{|E|} \tag{1}$$

where we summed over all error configurations E that make a logical error under the assumption of the optimal decoder. Here, $|E|$ is the weight of the error. We then change the expression to

$$P_{\text{fail}} = \sum_w N(w)(1-p)^{n-w} p^w \tag{2}$$

where $N(w)$ is the number of error configurations the weight of which is given by w (and yield a logical error).

Then let us naively (over)-count the number of error configurations that may produce a logical error. A logical X error is made when the recovering operator suggested by the decoder produces a connected path over a torus with the original errors. For a given logical operator of weight l in the toric code, one can show that the optimal decoder *may* choose such a path only if there are more than $l/2$ errors along the path. On the other hand, if there does not exist a connected path along the torus that contains more than $l/2$ erroneous qubits then the optimal decoder does not make a logical error from that error configuration (See also the discussion from the previous solution sheet). So when the decoder made a path of length l logical operator then we know that there were more than $l/2$ errors along that path. Then the question is that how many there are such error configurations. We obtain the upper bound of the number of configurations that satisfy the above constraints as follows. For each connected path (length l) along the torus, we consider an error configuration that contains $u > l/2$ errors on the path and v errors are not on the path. Then the weight of the total error is given by $u + v$. When there are $N_{\text{con}}(l)$ number of such paths, we obtain the upper bound for the number of such a configuration as

$$N_{\text{con}}(l) \times \binom{l}{u} \times \binom{n-l}{v}. \tag{3}$$

Using this, the desired result

$$P_{\text{fail}} \leq \sum_{l=d}^{n} N_{\text{con}}(l) \sum_{u=l/2}^{l} \sum_{v=0}^{n-l} \binom{l}{u} \binom{n-l}{v} \big(\frac{p}{1-p}\big)^{u+v} \tag{4}$$

is obtained.

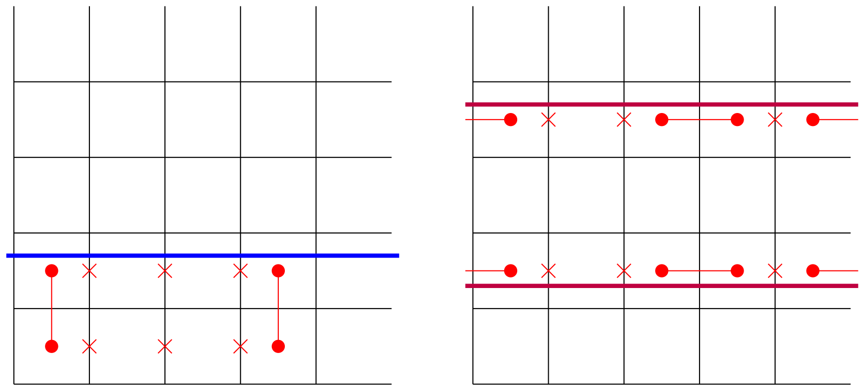

Figure 1: Left: As there is a loop that contains 3 errors, this error configuration is counted in Eq. (3). However, the optimal decoder suggests the red lines so does not yield a logical error. Right: Another possible error configuration that is counted but does not make a logical error. Even though when the optimal decoder choose a path that makes a loop (red path), it makes two logically equivalent operators (purple lines) so does not change the state. Moreover, this configuration is counted twice in Eq. (3).

Note that this is a rough upper bound because of two main reasons. First, there is an error configuration that does not make an logical error but counted. Even though there is a connected path along the torus that intersects with more than $l/2$ errors in a given error configuration, there may be a shorter path that does not make a logical error. Then the optimal decoder will choose the latter and error will be recovered. Another possible case is that even though the optimal decoder suggest a path along the line, if there are two such a path then they do not make a logical error. These two possibilities are depicted in Fig. 1. Second, if there are more than two loops that intersect with the errors with more than a half of the length, such error configuration is counted twice. The right example of Fig. 1 also show such an example.

Solution to Exercise 1.1
Using the above result, we can obtain an upper bound using

$$\sum_{v=0}^{n-l}\binom{n-l}{v}\Big(\frac{p}{1-p}\Big)^{v} = \Big(1+\frac{p}{1-p}\Big)^{n-l} = (1-p)^{n-l} \tag{5}$$

and

$$\sum_{u=l/2}^{l}\binom{l}{u}\Big(\frac{p}{1-p}\Big)^{u} \le \sum_{u=l/2}^{l}\binom{l}{u}\Big(\frac{p}{1-p}\Big)^{l/2} \le 2^{l}\Big(\frac{p}{1-p}\Big)^{l/2} \tag{6}$$

where the first inequality follows from $p/(1-p)\le 1$ for $p\le 1/2$ and the second one does from $\sum_{u=l/2}^{l}\binom{n}{l} \le \sum_{u=0}^{l}\binom{n}{l} = 2^{l}$. Then the result is

$$P_{\text{fail}} \le \sum_{l=d}^{n} N_{\text{con}}(l) 2^{l} p^{l/2}(1-p)^{l/2}. \tag{7}$$

Putting $N_{\text{con}}(l) < N_0 c^{l}$ yields

$$P_{\text{fail}} \le \sum_{l=d}^{n} N_0 (2c\sqrt{p(1-p)})^{l}. \tag{8}$$

In the asymptotic limit ($d \gg 1$), the summation converges only when $(2c\sqrt{p(1-p)}) < 1$ and the summation is dominated by the leading term $N_0(2c\sqrt{p(1-p)})^d$ so approximated by $\approx e^{-d/\xi}$ that gives the threshold. Solving this inequality gives the condition $p < p_{\mathrm{th}} = 0.0373$.