14756: Quantum Error Correction **WS 2018**

Solution set 7

Tutor: Chae-Yeun Park *Dec. 10*

Excercise 1

Solution to Exercise 1.1
Define $\tilde{P}_{\mathcal{C}} = \sum_k |\psi_k\rangle \langle\psi_k|$ where $\{|\psi_k\rangle\}$ are orthonormal basis vectors of $\mathcal{C}$. Then let us first show that $\tilde{P}_{\mathcal{C}} = P_{\mathcal{C}} := \Pi_j P_j$. First, note that possible eigenvalues of both projectors are $+1$ and 0. Then eigensubspace of $P_{\mathcal{C}}$ with eigenvalue $+1$ is $\mathcal{C} = \{|\psi\rangle \, |P_j |\psi\rangle = |\psi\rangle \, \forall j\}$ by the definition. As it is also obvious that eigensubspace $\tilde{P}_{\mathcal{C}}$ with eigenvalue $+1$ is C, we see that both projectors are the same. Then

$$P_{\mathcal{C}} O P_{\mathcal{C}} = \sum_{k,l} |\psi_k\rangle \langle\psi_k| O |\psi_l\rangle \langle\psi_l| = \sum_{k,l} |\psi_k\rangle \langle\psi_l| \langle\psi_k|O|\psi_l\rangle \tag{1}$$

$$= \sum_{k,l} |\psi_k\rangle \langle\psi_l| \delta_{kl} c(O) = c(O) P_{\mathcal{C}} \tag{2}$$

where $c(O) = \langle\psi_k|O|\psi_k\rangle$ that is independent to k by the assumption.

Solution to Exercise 1.2
Define $dist(j, A)$ be the minimum distance between the qubit j and the region A. Then we have

$$P_{\mathcal{C}} O_A O_B P_{\mathcal{C}} = P_{\mathcal{C}} \Pi_{dist(j,A)>r} P_j O_A O_B \Pi_{dist(j,B)>r} P_j P_{\mathcal{C}} \tag{3}$$

$$= P_{\mathcal{C}} O_A \Pi_{dist(j,A)>r} P_j \Pi_{dist(j,B)>r} P_j O_B P_{\mathcal{C}} \tag{4}$$

$$= P_{\mathcal{C}} O_A P_{\mathcal{C}} O_B P_{\mathcal{C}} \tag{5}$$

$$= P_{\mathcal{C}} O_A P_{\mathcal{C}} P_{\mathcal{C}} O_B P_{\mathcal{C}} \tag{6}$$

$$= c(O_A) c(O_B) P_{\mathcal{C}} \tag{7}$$

where we have used $P_j^2 = P_j$ and $P_{\mathcal{C}} = \Pi_j P_j$ in the first line, P_j that does not acts on A commutes with A in the second line, and $\Pi_{dist(j,A)>r} P_j \Pi_{dist(j,B)>r} P_j = P_{\mathcal{C}}$ in the third line as $\{dist(j, B) > r\} \cup \{dist(j, A) > r\}$ is the whole lattice if $dist(A, B) > r$).

Solution to Exercise 1.3
Let us recall the original proof in 2D lattice. First, holographic disentangling lemma is used to choose a $R \times R$ square that is correctable with the largest R. The lemma states that for given A that is correctable, if there is correctible B that contains the boundary of A then $A \cup B$ is also correctible. Thus we first pick a square with d qubits, that is correctible by the definition of the code, and add the boundary of it. We can continue this procedure while the boundary of A contains qubits less than d. As the boundary of $R \times R$ square proportional to R, the termination condition is given by $R \sim d$. After dividing the whole lattice to n/R^2 squares (where n is the number of qubits in the lattice), we punch the circle (with radius r) in the intersecting region. Then we may use the above exercise to show that the lattice can be divided by three regions like the figure with correctible A and B.

After this construction, we use another lemma derived in the lecture that $k = S(\Lambda) = S(C) \leq |C|$. From the construction, we can count the number of qubits in C. As there are n/R^2 intersecting points, we know $|C| \sim n/R^2$ and $R \sim d$. This is how we obtain $kd^2/n \leq c$.

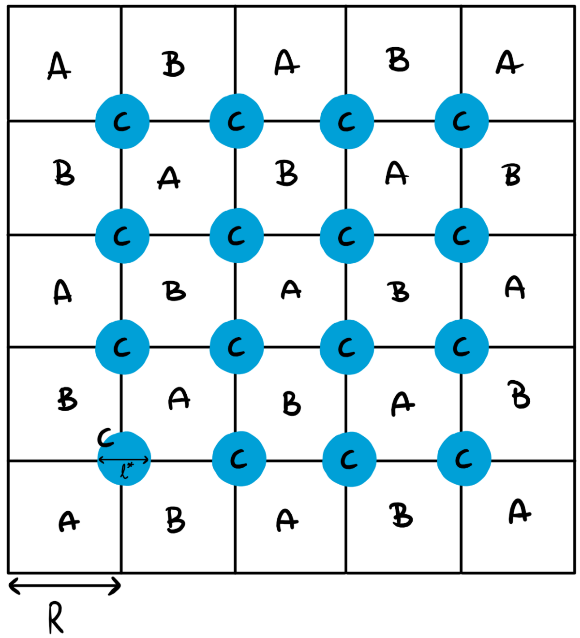

Figure 1: 2D lattice divided by correctible regions.

In case of D dimensional lattice, we use the same lemma to construct the hypercubes that covers the lattice. As the boundary of a hypercube is given by $D-1$ dimensional surface, we have $R^{D-1} \sim d$. After dividing the whole lattice into hypercubes, we label each neighboring hypercube A and B in checkerboard manner. Then hypercubes in A intersect in $D-2$ dimensional face. We have to punch this area and insert C to make the cubes be separated by r. This induces the condition of C that $|C| \sim R^{D-2} \times n/R^D = n/R^2$ as there are n/R^D faces and $\sim R^{D-2}$ qubits are inserted to each face. To sum up, we have $R^{D-1} \sim d$ and $|C| \sim n/R^2$. Using the same entropic inequality, we have $kd^{2/(D-1)}/n \leq c$.