**Exercise 1**

**Solution to Exercise 1.1**

Define $\tilde{P}_C = \sum_k |\psi_k\rangle \langle \psi_k|$ where $\{|\psi_k\rangle\}$ are orthonormal basis vectors of $C$. Then let us first show that $\tilde{P}_C = P_C := \Pi_j P_j$. First, note that possible eigenvalues of both projectors are $+1$ and $0$. Then eigensubspace of $P_C$ with eigenvalue $+1$ is $C = \{|\psi\rangle | P_j |\psi\rangle = |\psi\rangle \forall j\}$ by the definition. As it is also obvious that eigensubspace $\tilde{P}_C$ with eigenvalue $+1$ is $C$, we see that both projectors are the same. Then

$$P_C O P_C = \sum_{k,l} |\psi_k\rangle \langle O |\psi_l\rangle = \sum_{k,l} |\psi_k\rangle \langle \psi_l\rangle |O|\psi_k\rangle (1)$$

$$= \sum_{k,l} |\psi_k\rangle \langle \psi_l\rangle \delta_{kl} c(O) = c(O) P_C (2)$$

where $c(O) = \langle \psi_k|O|\psi_k\rangle$ that is independent to $k$ by the assumption.

**Solution to Exercise 1.2**

Define $\text{dist}(j,A)$ be the minimum distance between the qubit $j$ and the region $A$. Then we have

$$P_C O_A O_B P_C = P_C \Pi_{\text{dist}(j,A) > r} P_j O_A O_B \Pi_{\text{dist}(j,B) > r} P_j P_C (3)$$

$$= P_C O_A \Pi_{\text{dist}(j,A) > r} P_j \Pi_{\text{dist}(j,B) > r} P_j O_B P_C (4)$$

$$= P_C O_A P_C O_B P_C (5)$$

$$= P_C O_A P_C P_C O_B P_C (6)$$

$$= c(O_A) c(O_B) P_C (7)$$

where we have used $P_j^2 = P_j$ and $P_C = \Pi_j P_j$ in the first line, $P_j$ that does not acts on $A$ commutes with $A$ in the second line, and $\Pi_{\text{dist}(j,A) > r} P_j \Pi_{\text{dist}(j,B) > r} P_j = P_C$ in the third line as $\{\text{dist}(j,B) > r\} \cup \{\text{dist}(j,A) > r\}$ is the whole lattice if $\text{dist}(A,B) > r$.

**Solution to Exercise 1.3**

Let us recall the original proof in 2D lattice. First, holographic disentangling lemma is used to choose a $R \times R$ square that is correctable with the largest $R$. The lemma states that for given $A$ that is correctable, if there is correctible $B$ that contains the boundary of $A$ then $A \cup B$ is also correctible. Thus we first pick a square with $d$ qubits, that is correctable by the definition of the code, and add the boundary of it. We can continue this procedure while the boundary of $A$ contains qubits less than $d$. As the boundary of $R \times R$ square proportional to $R$, the termination condition is given by $R \sim d$. After dividing the whole lattice to $n/R^2$ squares (where $n$ is the number of qubits in the lattice), we punch the circle (with radius $r$) in the intersecting region. Then we may use the above exercise to show that the lattice can be divided by three regions like the figure with correctible $A$ and $B$.

After this construction, we use another lemma derived in the lecture that $k = S(A) = S(C) \leq |C|$. From the construction, we can count the number of qubits in $C$. As there are $n/R^2$ intersecting points, we know $|C| \sim n/R^2$ and $R \sim d$. This is how we obtain $kd^2/n \leq c$. 

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**14756: Quantum Error Correction**

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**Solution set 7**

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In case of $D$ dimensional lattice, we use the same lemma to construct the hypercubes that covers the lattice. As the boundary of a hypercube is given by $D - 1$ dimensional surface, we have $R^{D-1} \sim d$. After dividing the whole lattice into hypercubes, we label each neighboring hypercube $A$ and $B$ in checkerboard manner. Then hypercubes in $A$ intersect in $D - 2$ dimensional face. We have to punch this area and insert $C$ to make the cubes be separated by $r$. This induces the condition of $C$ that $|C| \sim R^{D-2} \times n/R^D = n/R^2$ as there are $n/R^D$ faces and $\sim R^{D-2}$ qubits are inserted to each face. To sum up, we have $R^{D-1} \sim d$ and $|C| \sim n/R^2$. Using the same entropic inequality, we have $kd^{D/(D-1)}/n \leq c$. 

Figure 1: 2D lattice divided by correctible regions.