Exercise Sheet 2

Kastoryano: Quantum Error Correction

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1 Exercise 1: Mixed state and the Bloch sphere

Exercise 1.1 (from N&C) Let ρ be a density operator on $\mathcal{H}_d$, for some finite $d \geq 2$. Show that $\mathrm{tr}[\rho^2] \leq 1$, with equality if and only if ρ is pure.

Exercise 1.2 (from N&C)

1. Show that an arbitrary density matrix for a mixed qubit state can be written as
$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}), \tag{1}$$
where $\vec{r}$ is a three dimensional vector such that $||\vec{r}|| \leq 1$, and $\vec{\sigma} = (X, Y, Z)$ is the vector of Pauli matrices.

2. What is the Bloch vector representation of $\rho = \alpha|+\rangle\langle+| + (1-\alpha)|-\rangle\langle-|$ for $0 \leq \alpha \leq 1$?

3. Show that ρ is pure if and only if $||\vec{r}|| = 1$.

2 Exercise 2: Quantum Measurements

Exercise 2.1 (from N&C) Let $\vec{u}$ be a three dimensional unit vector (i.e. $||\vec{u}|| = 1$). Show that $\vec{u} \cdot \vec{\sigma}$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspacees are given by $P_\pm = \frac{1}{2}(\mathbb{1} \pm \vec{u} \cdot \vec{\sigma})$?

Exercise 2.2 (from N&C) Calculate the probability of obtaining the result $+1$ for a measurement of $\vec{u} \cdot \vec{\sigma}$ given that the state prior to measurement is $|0\rangle$. What is the state of the system after the measurement if $+1$ is obtained?

3 Exercise 3: Quantum channels

A map T taking the space of $d \times d$ matrices M_d to itself is said to be *positive* if $T(X) \geq 0$ whenever $X \geq 0$, with $X \in \mathcal{M}_d$. The map is *completely positive* if $\mathrm{id} \otimes T(X) \geq 0$, whenever $X \geq 0$ for $X : \mathcal{M}_{d^2} \to \mathcal{M}_{d^2}$. Show that the transpose operation is positive but not completely positive.

(Hint: consider a two qubit maximally entangled state $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.)