1 Exercise 1: Lower bound on the threshold

Exercise 1.1 : Argue (as I did in class) that the logical failure of a toric code of length \( L \) with physical error rate \( p \) is upper bounded by the expression:

\[
P_{\text{fail}}(n, p) \leq (1 - p)^n \sum_{l=d}^{n} N_{\text{con}}(l) \sum_{u=l/2}^{l} \sum_{v=0}^{n-l} \binom{l}{u} \binom{n-l}{v} \left( \frac{p}{1-p} \right)^{u+v},
\]

where \( N_{\text{con}} \) is the number of length-\( l \) non-contractible closed loops on the torus, constrained by the requirement that they can have no self-intersections.

Recall: \( \binom{a}{b} = \frac{a!}{b!(a-b)!} \) is the binomial coefficient.

Exercise 1.2 Show that this expression can be upper bounded as

\[
P_{\text{fail}} \leq \sum_{l=d}^{n} N_{\text{con}}(l) 2^l p^{l/2} (1 - p)^{l/2}
\]

\[
\leq \sum_{l=d}^{n} N_0(2c\sqrt{p(1-p)})^l,
\]

where we use that \( N_{\text{con}}(l) < N_0 c^l \), for some constant \( N_0 \) and the lattice expansion coefficient \( c \approx 2.638 \). What value does this lower bound give on the threshold?