QUANTUM GIBBS SAMPLERS: THE COMMUTATIVE CASE

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MOTIVATION

Simulation of systems in thermal equilibrium

Analysis of thermalization in nature

Can we say anything about the difficulty of simulating a state, just from the state?

Does nature always prepare "easy states" efficiently?

MOTIVATION

Main structural theorem:

Rate of convergence

$\Leftrightarrow$

Correlations in Gibbs state

Characterizes the thermodynamically trivial phase

SETTING

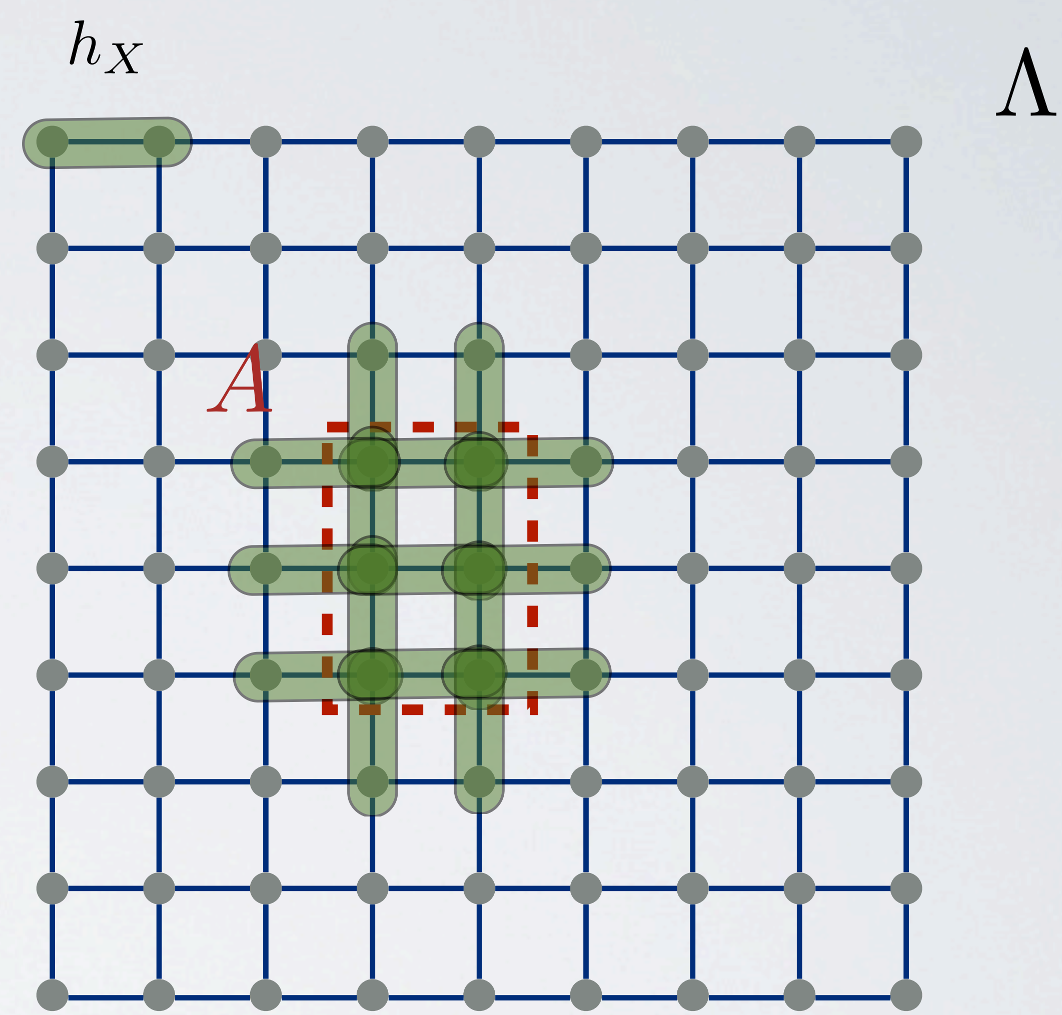
Finite lattice system

Finite local dimension

Bounded, local and **commuting** Hamiltonian

$$H_A = \sum_{Z \in A} h_Z \qquad [h_Z, h_Y] = 0, \quad \forall Z, Y$$

$$A \subset \Lambda$$



Global Gibbs state: $\rho \propto e^{-\beta H_\Lambda}$

Non-commutative $\mathbb{L}_p$ spaces:

$$\langle f, g \rangle_\rho = \mathrm{tr}[\rho^{1/2} f^\dagger \rho^{1/2} g]$$ $\mathbb{L}_p$ inner product

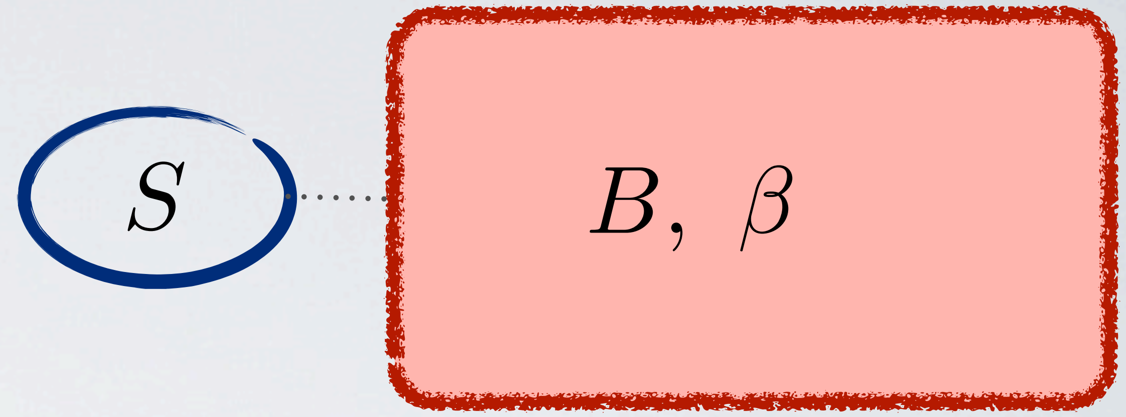
$$||f||_{p,\rho}^p = \mathrm{tr}[|\rho^{1/2p} f \rho^{1/2p}|^p]$$ $\mathbb{L}_p$ norm

Def: Gibbs samplers are primitive semigroups with Gibbs state as unique stationary state

GIBBS SAMPLERS

Davies generators

Finite system weakly coupled
to a markovian thermal bath



Bath autocorolation fcn

$$\mathcal{L}_A(f) = \sum_{\alpha(j),j\in A,\omega} g_{\alpha(j)}(\omega)(S_{\alpha(j)}(\omega) f S^{\dagger}_{\alpha(j)}(\omega) - \frac{1}{2}\{S_{\alpha(j)}(\omega) S^{\dagger}_{\alpha(j)}(\omega)\})$$

$A \subset \Lambda$

Jump operators: between
eigenstates of H

Properties: Completely positive Local (same locality as H)

Locally reversible: $\langle f, \mathcal{L}_A(g)\rangle_\rho = \langle \mathcal{L}_A(f), g\rangle_\rho$

GIBBS SAMPLERS

Heat-bath generators

local projection onto Gibbs state

$$\mathcal{L}_A(f) = \sum_{k\in A} \mathbb{E}_k^{\rho}(f) - f$$

$$\mathbb{E}_k^{\rho}(f) = \mathrm{tr}_k[\gamma_k f \gamma_k^{\dagger}]$$

$\mathbb{E}_A^{\rho}$ is a conditional expectation

$$\gamma_k = (\mathrm{tr}_k[\rho])^{-1/2}\rho^{1/2}$$

Only depends on properties of the state.

Properties: Completely positive Local (same locality as H)

Locally reversible: $\langle f, \mathcal{L}_A(g)\rangle_\rho = \langle \mathcal{L}_A(f), g\rangle_\rho$

RELAXATION TIME

We want to estimate how rapidly the sampler converges to the Gibbs state

Trace norm bound: $||e^{t\mathcal{L}}(\phi) - \rho||_1 \leq \epsilon$

Mixing time: $\tau \geq \frac{\log(||\rho^{-1}||/\epsilon)}{\lambda}$

$||\rho^{-1}|| \leq e^{o(|\Lambda|)}$

$$\tau \propto |\Lambda|/\lambda_\Lambda$$

Reduces to estimating the gap!

$$\boxed{\lambda_A = \inf_{f \in \mathcal{A}_\Lambda} \frac{\langle f, -\mathcal{L}_A(f)\rangle_\rho}{\mathrm{Var}_A(f)}}$$

where $\mathrm{Var}_A = ||f - \mathbb{E}^\rho_A(f)||^2_{2,\rho}$

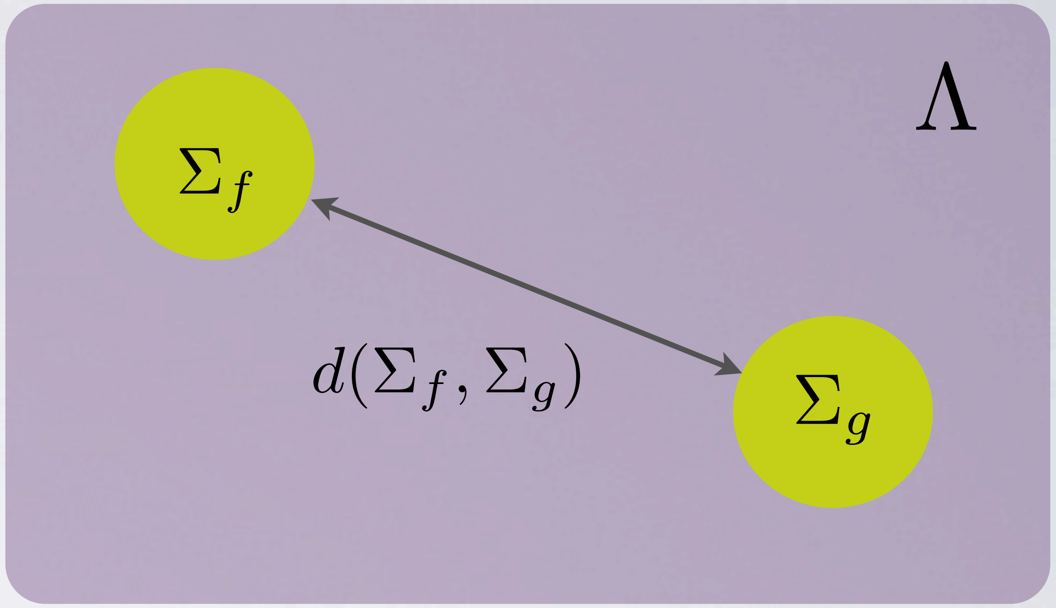
CLUSTERING

Def: weak clustering

different norm

$$\mathrm{Cov}(f,g) \leq c||f||_{2,\rho}||g||_{2,\rho}e^{-d(\Sigma_f,\Sigma_g)/\xi}$$

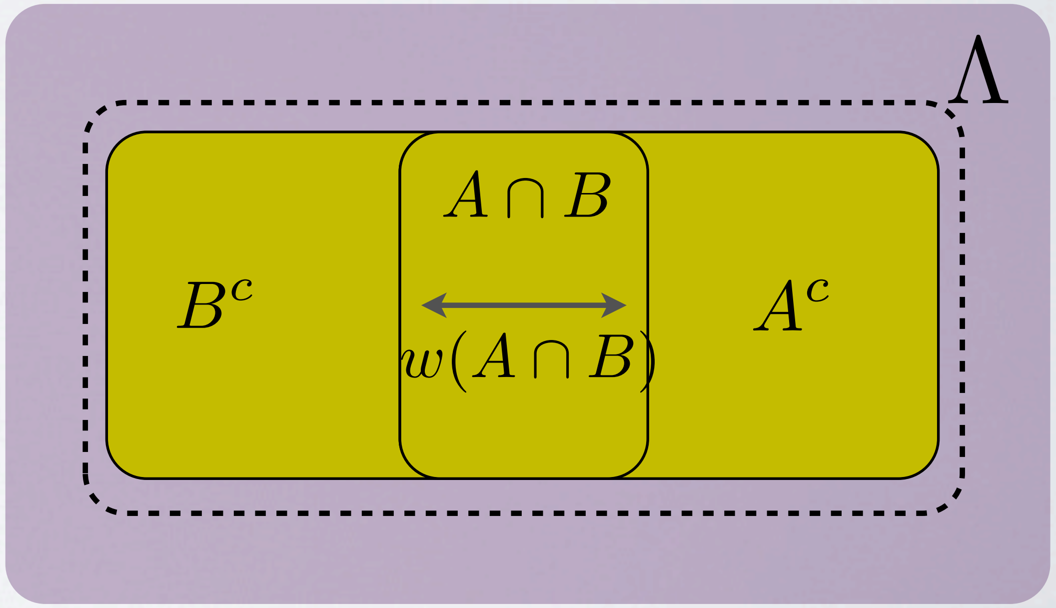
$$\mathrm{Cov}(f,g) = \langle f - \langle f\rangle_\rho, g - \langle g\rangle_\rho\rangle_\rho$$



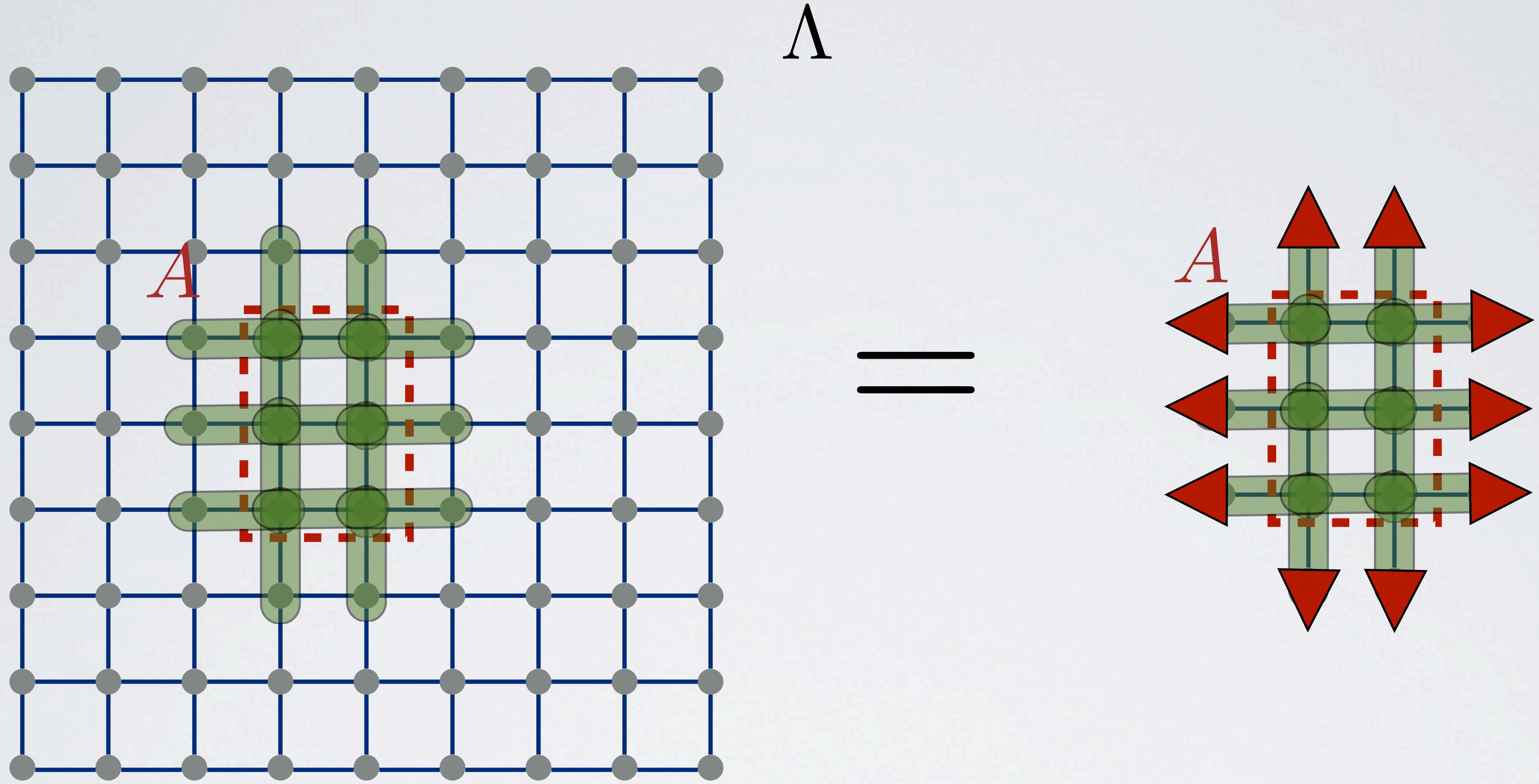
Def: strong clustering

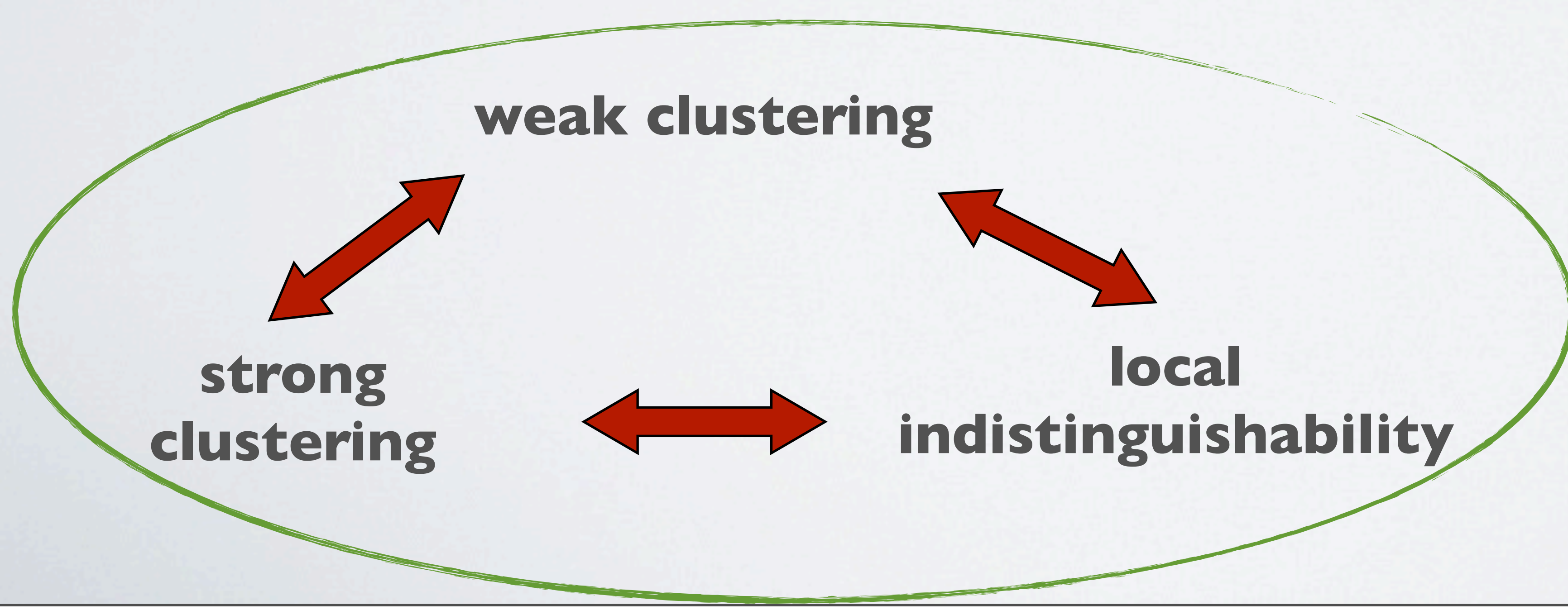
$$\mathrm{Cov}_{A\cup B}(\mathbb{E}_A(f),\mathbb{E}_B(f)) \leq c||f||_{2,\rho}^2 e^{-w(A\cap B)/\xi}$$

$$\mathrm{Cov}_{A\cup B}(f,g) = \langle f - \mathbb{E}_A(f), g - \mathbb{E}_B(g)\rangle_\rho$$



DLR THEORY (CLASSICAL)





Equivalence breaks down for quantum systems!

MAIN THEOREM

$\mathcal{L}$ **is gapped**

$\Longleftrightarrow$

$\mathcal{L}$ **satisfies strong clustering**

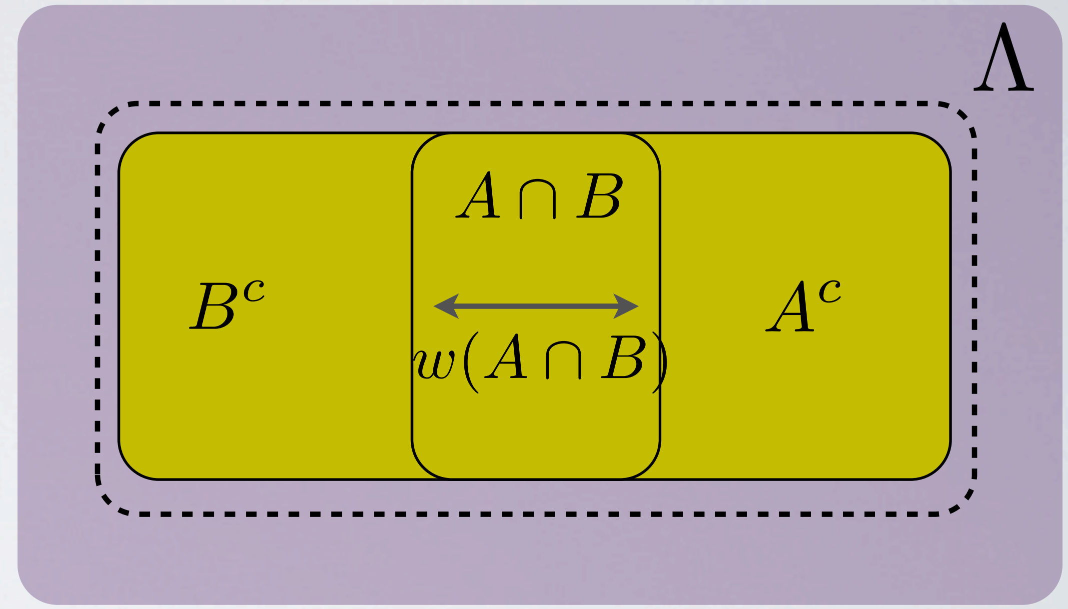
PROOF OUTLINE

Prop: If the Gibbs state satisfies strong clustering,

$$\mathrm{Cov}_{A\cup B}(\mathbb{E}_A(f), \mathbb{E}_B(f)) \leq \epsilon ||f||^2_{2,\rho}$$

then

$$\mathrm{Var}_{A\cup B}(f) \leq (1+\epsilon)(\mathrm{Var}_A(f) + \mathrm{Var}_B(f))$$



Assume $\quad w(A\cap B) \approx \sqrt{L} \qquad w(A) \approx w(B) \approx L$

can eliminate this term by averaging

$$\begin{aligned}
\mathrm{Var}_{A\cup B}(f) &\leq (1+\epsilon)(\mathrm{Var}_A(f) + \mathrm{Var}_B(f)) \\
&\leq (1+\epsilon)(\lambda_A^{-1}\langle f, -\mathcal{L}_A(f)\rangle_\rho + \lambda_B^{-1}\langle f, -\mathcal{L}_B(f)\rangle_\rho) \\
&\leq (1+\epsilon)\lambda_{A,B}^{-1}(\langle f, -\mathcal{L}_{A\cup B}(f)\rangle_\rho + \langle f, -\mathcal{L}_{A\cap B}(f)\rangle_\rho)
\end{aligned}$$

Thus we get: $\quad \lambda(2L) \approx \lambda(L) \quad$ since $\quad \epsilon \leq c e^{-\sqrt{L}/\xi}$

applying iteratively completes the proof

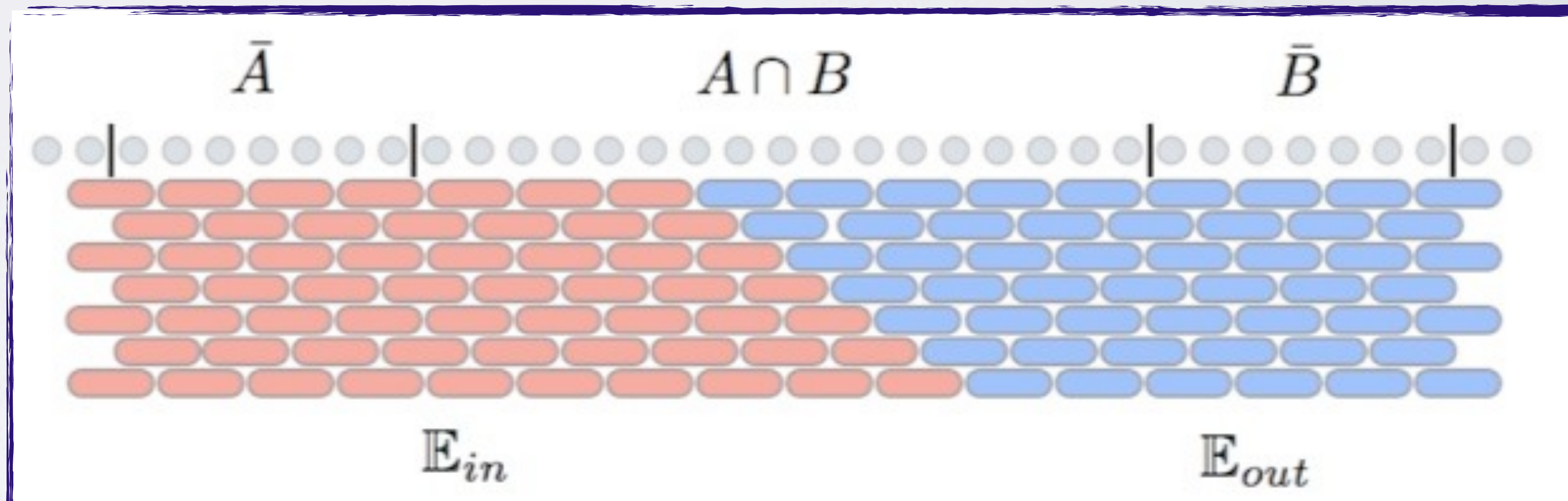
PROOF OUTLINE

Map Liouvillian onto FF Hamiltonian

	Commuting Gibbs Sampler	Frustration-free Hamiltonian
State	Gibbs state ρ	Ground state $\|\varphi\rangle$
Dynamics	Reversible Liouvillians $\mathcal{L}$	Hamiltonian H
Projectors	Conditional Expectations $\mathbb{E}$	Ground state projectors P
Gap	Spectral gap of $\mathcal{L}$	Spectral gap of H
Framework	$\mathbb{L}_p$ spaces	Hilbert spaces $\mathcal{H}$

Use the detectability lemma

Can invoke the theory of FF gaped Hamiltonians



By constructing an approximate projector

$$\Pi^l \approx \mathbb{E} = \mathbb{E}_{in}\mathbb{E}_{out}$$

it is not difficult to show that

$$||\hat{\mathbb{E}}_A\hat{\mathbb{E}}_B - \hat{\mathbb{E}}_{A\cup B}|| \leq e^{-l\lambda/\xi}$$

MAIN THEOREM

$\mathcal{L}$ **is gapped**

$\Updownarrow$

$\mathcal{L}$ **satisfies strong clustering**

APPLICATIONS

In 1D strong and weak clustering are equivalent

Boundaries can be removed in 1D

In 1D Gibbs samplers are always gapped

One can use MPS methods in 1D

Beyond a universal critical temperature Gibbs samplers are gapped

Note: cannot use Araki's result!

OUTLOOK

Consider what this means for topological order at non-zero temperature

Extend the results to get Log-Sobolev bounds

What can we say about the non-commuting case?

THANK YOU FOR YOUR ATTENTION!