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LOCAL APPROXIMATE ERROR CORRECTION CODES

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MOTIVATION

Intriguing example

Crépeau et. al. (2005), quant-ph/0503139

- No quantum code can correct more than $n/4$ arbitrary errors → Consequence of no-cloning theorem

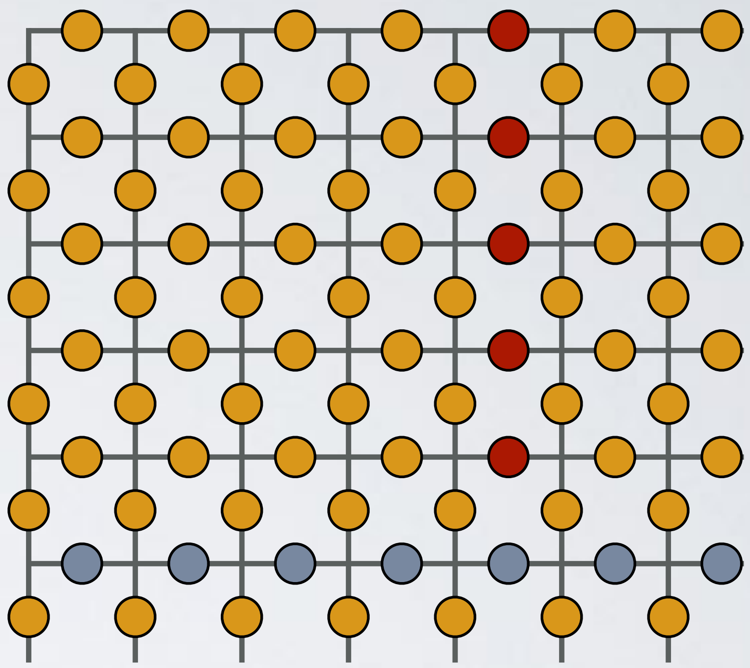
- Classical codes (Ex: repetition code) can correct up to $\lfloor n/2 \rfloor$ arbitrary classical errors

Crépeau et. al. (2005) construct an approximate quantum code that can correct up to $\lfloor n/2 \rfloor$ arbitrary quantum errors!

→ Indication that approximate codes can outperform exact codes!

MOTIVATION

What about topological codes?



- Codes often characterised by three numbers: length n ; distance d ; encoded (qu-)bits k

Tradeoff bounds

- $kd^2 \leq cn$ Commuting projector codes
  Bravyi, Poulin, Terhal
- $kd \leq cn$ Subsystem codes
  Bravyi
- $kd^{1/2} \leq cn$ Classical lattice systems
  Bravyi, Poulin, Terhal; Yoshida

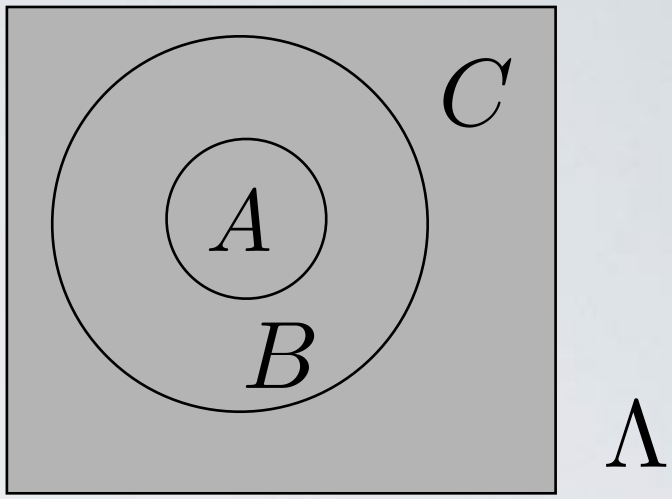
Where do approximate quantum codes sit?

Lattice commuting projector codes

$$\{S_j\} \quad [S_j, S_k] = 0 \quad S_j = S_j^2$$

$$\Pi = \prod_j S_j \quad \mathcal{C} = \{|\psi\rangle, \Pi|\psi\rangle = |\psi\rangle\}$$

➡ $\mathcal{C}$ is the codespace ➡ Erasure errors



Lattice commuting projector codes

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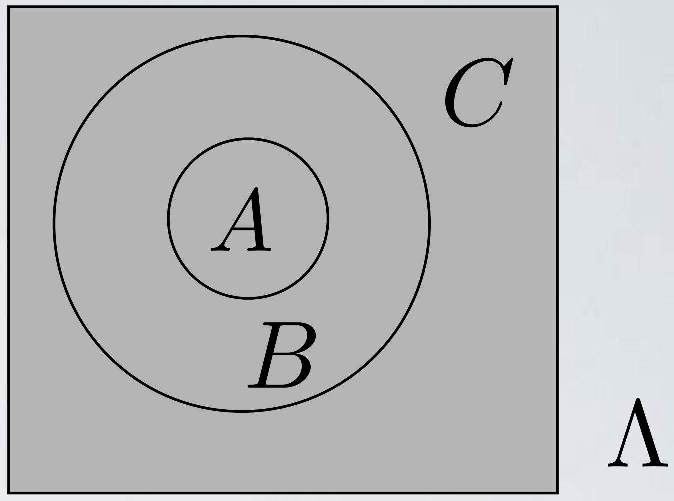

- $\mathcal{C}$ is the codespace
- Erasure errors

Lemma *Let $\mathcal{C}$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between A and C is at least $\ell \geq w$, the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:*

(i) *Topological Quantum Order (TQO): for any observable O_A with support on A, any two ground states $|\phi\rangle$ and $|\psi\rangle$ give the same expectation value, $\langle\phi| O^A |\phi\rangle = \langle\psi| O^A |\psi\rangle$.*

(ii) *Decoupling: For any $\rho \in \mathcal{C}$ we have $I_\rho(A : CR) = 0$.*

(iii) *Error correction: There exists a recovery map acting on AB such that $\mathcal{R}_B^{AB}(\rho^{BC}) = \rho^{ABC}$ for any $\rho \in \Pi$.*

(iv) *Disentangling unitary: For any $\rho \in \mathcal{C}$ there exists a unitary U^B, such that $U^B \rho U^{B\dagger} = \omega^{AB_1} \otimes \rho^{B_2 C}$, for some state ω^{AB_1}.*

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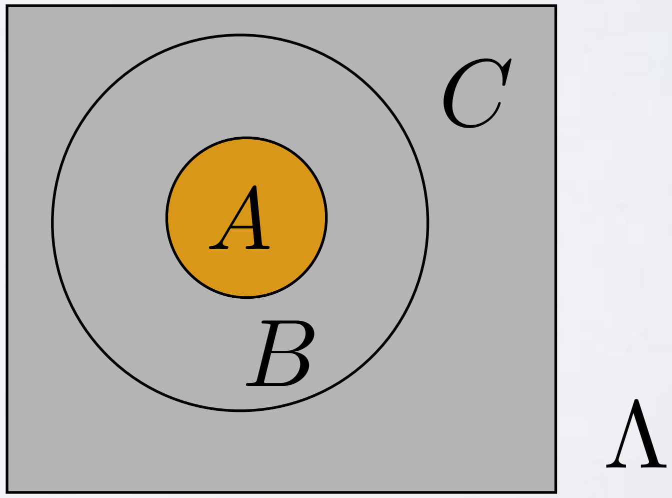
- (i) Topological order



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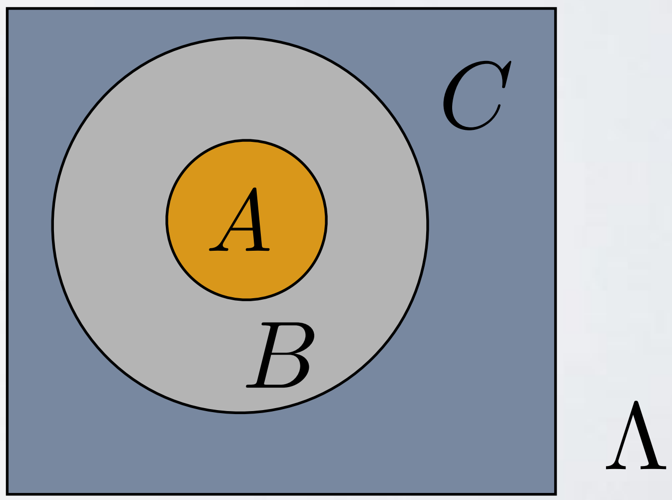
- (i) Topological order
- (ii) Decoupling $I_\rho(A:CR) = S(A) + S(AB) - S(B)$



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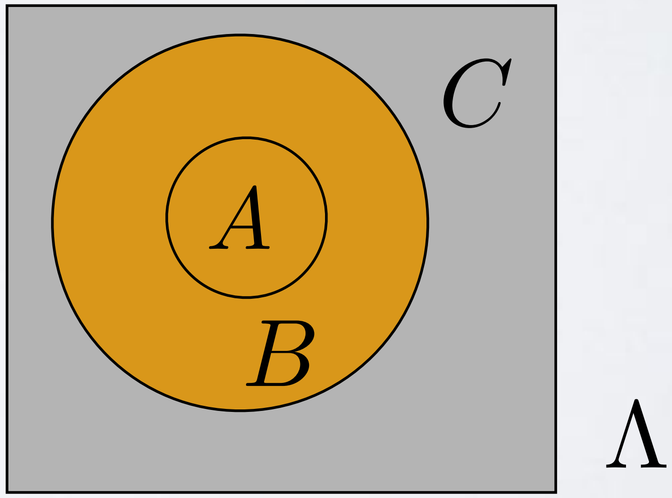
- (iii) Error correction



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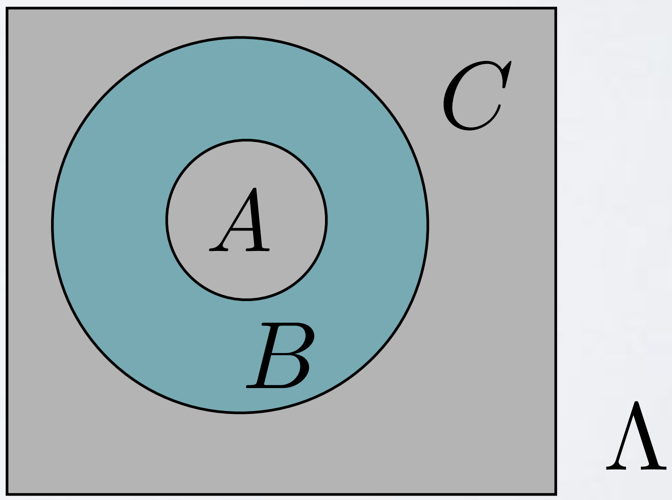
- (iii) Error correction
- (iv) Disentangling unitary



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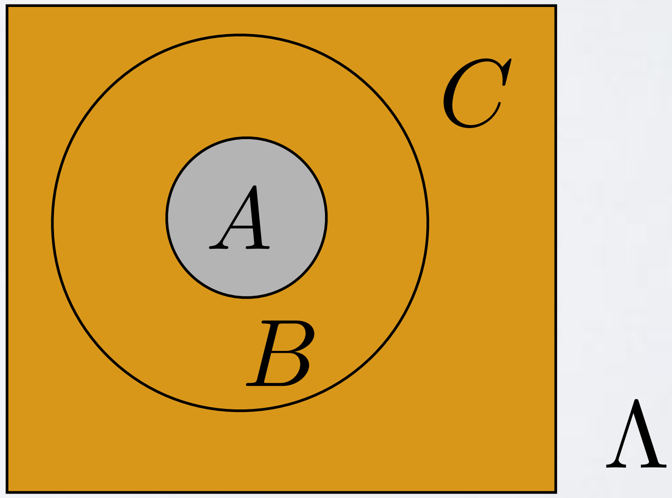
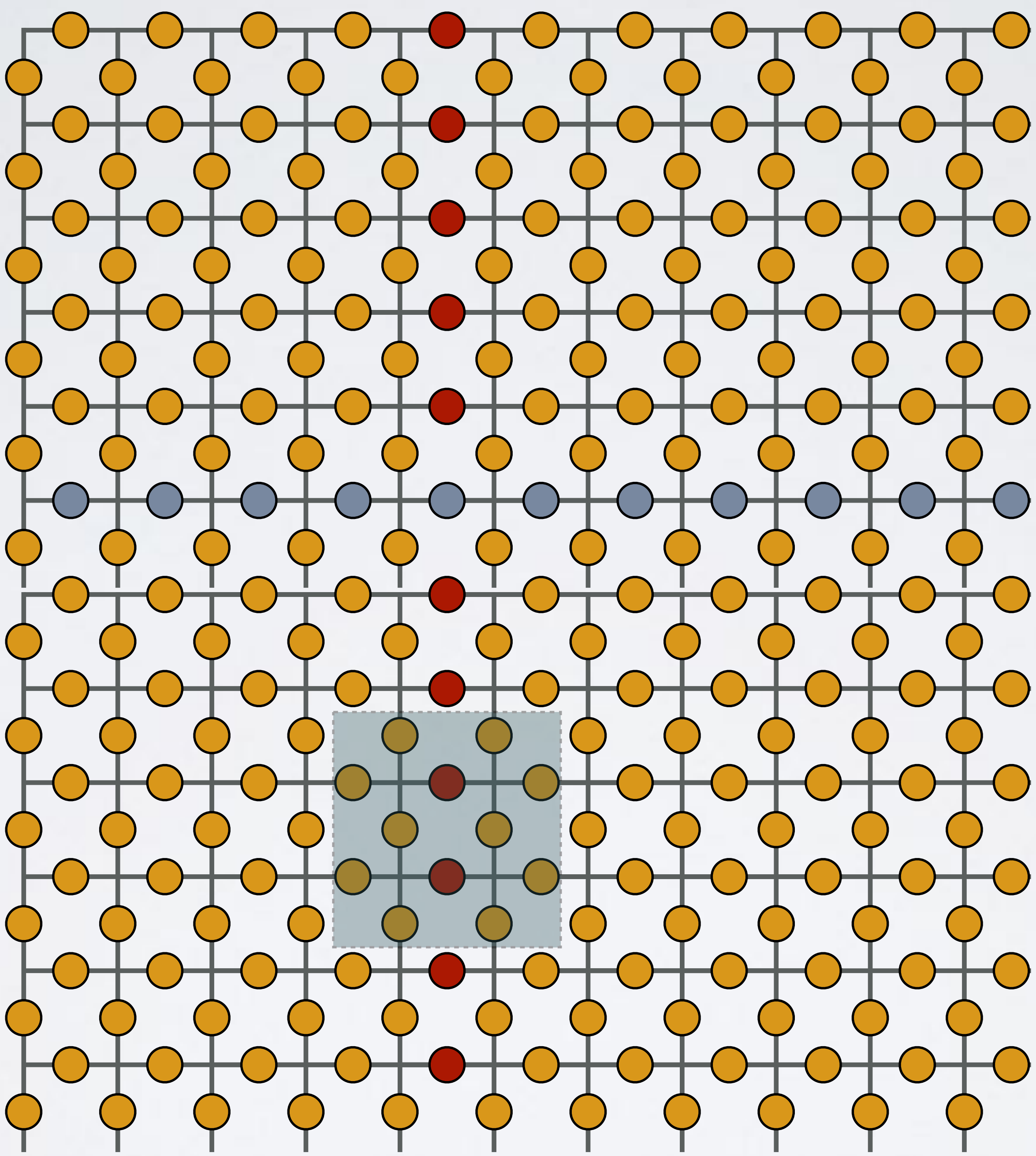
- (ii) Decoupling
- (iii) Error correction
- (iv) Disentangling unitary
- (v) Cleaning



CLEANABILITY



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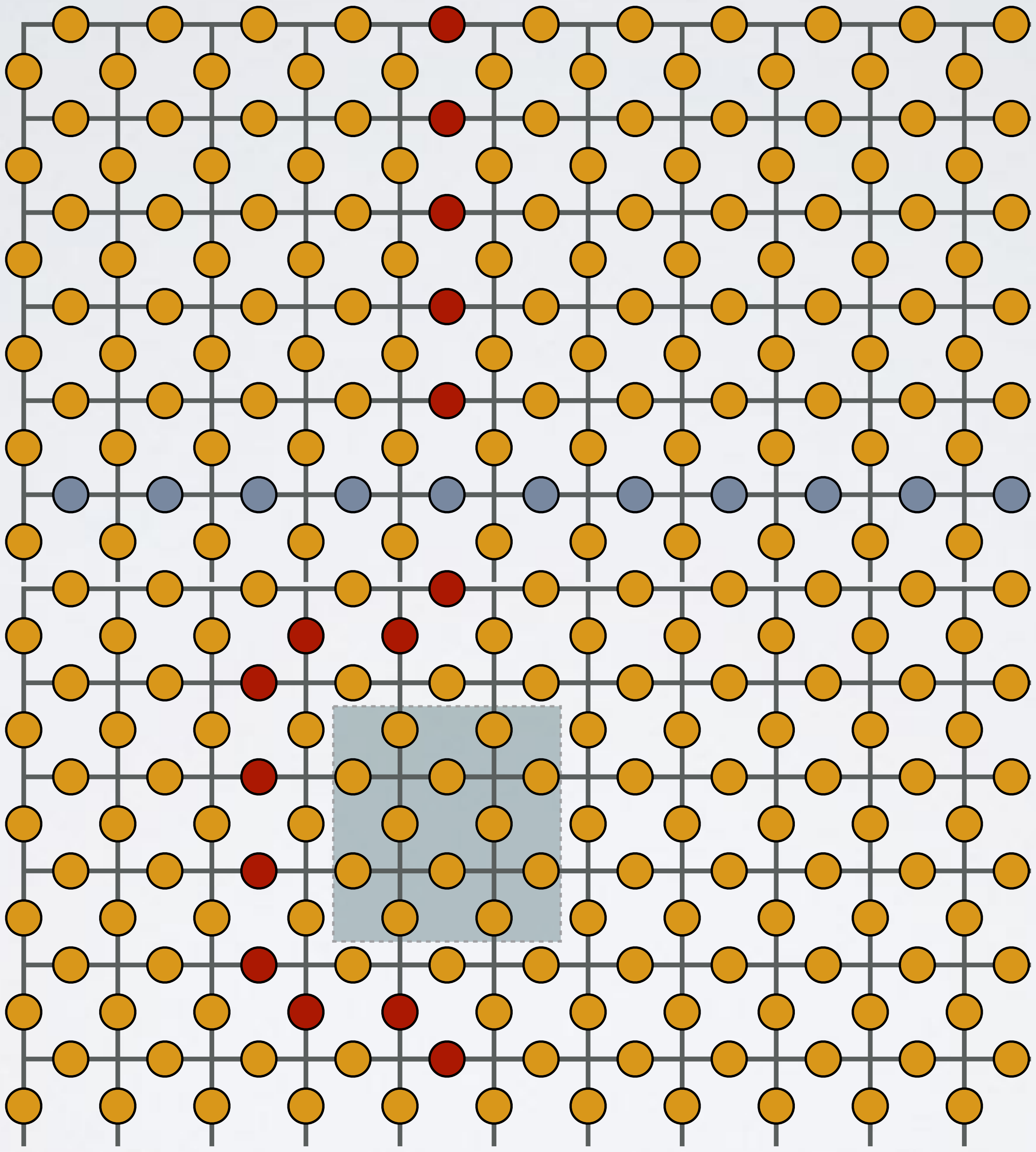

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Which properties can be extended to approximate codes?

➡ Focus on topological codes; tradeoff bounds

BPT BOUND?

Tradeoff bound

$$\boxed{kd^2 \leq cn}$$

Subspace or commuting projector codes

Bravyi, Poulin, Terhal

⇨ Toric code saturates the bound in 2D

Proof:

- Expansion bound
- Union bound
- Counting degrees of freedom

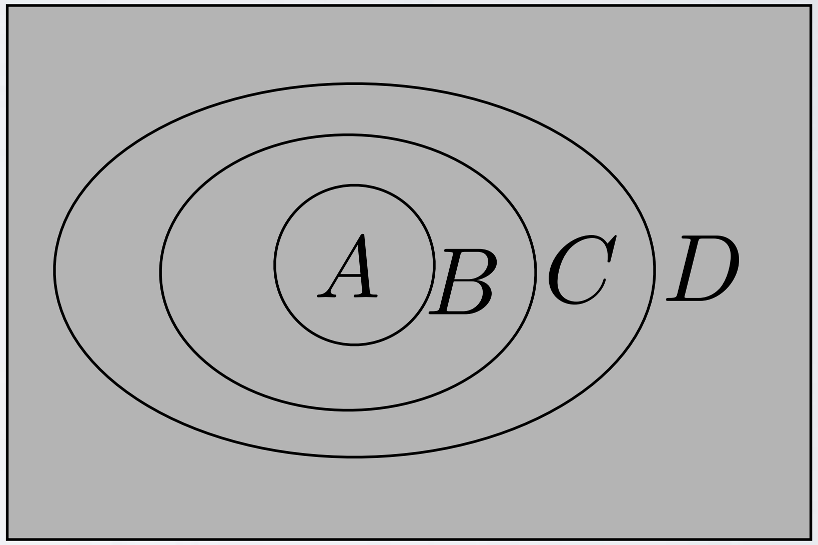
BPT BOUND?

$$\boxed{kd^2 \leq cn}$$

<u>Expansion Lemma:</u>

If A is correctable and B is correctible, then $A \cup B$ is correctable.

Proof:



Λ

A correctable $\Rightarrow \quad \rho^{ACD} = \omega^A \otimes \rho^{CD}$ (iv)

B correctable $\Rightarrow \mathcal{R}_{AC}^{ABC}(\rho^{ACD}) = \rho^{ABCD}$ (iii)

Define a map $\quad \mathcal{F}_C^{ABC}(\rho^{CD}) = \mathcal{R}_{AC}^{ABC}(\omega^A \otimes \rho^{CD})$

Show (iii) $\quad \mathcal{F}_C^{ABC}(\rho^{CD}) = \mathcal{R}_{AC}^{ABC}(\omega^A \otimes \rho^{CD}) = \mathcal{R}_{AC}^{ABC}(\rho^{ACD}) = \rho^{ABCD}$

$\square$

BPT BOUND?

$$\boxed{kd^2 \leq cn}$$

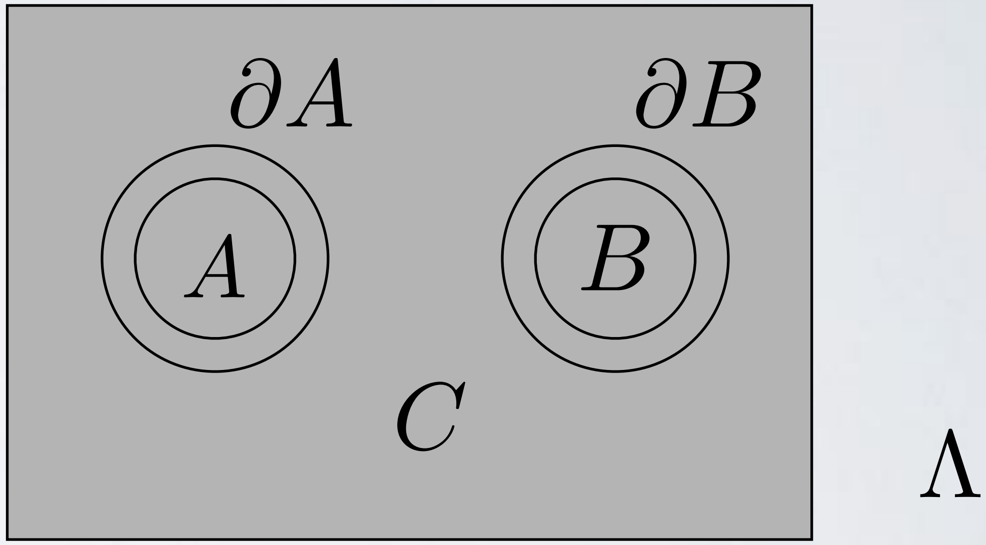
Union Lemma:

If A is correctable and B is correctible, then $A \cup B$ is correctable.

Proof:

A correctable $\Rightarrow \quad \mathcal{R}^{B\partial B}_{\partial B}(\rho^{\Lambda \setminus B}) = \rho^{\Lambda}$ (iv)

B correctable $\Rightarrow \quad \mathcal{R}^{B\partial B}_{\partial B}(\rho^{\Lambda \setminus A}) = \rho^{\Lambda}$ (iii)



Clearly, $\quad \mathcal{R}^{AB\partial B}_{\partial AB}(\rho^{\Lambda \setminus AB}) = \rho^{\Lambda}$

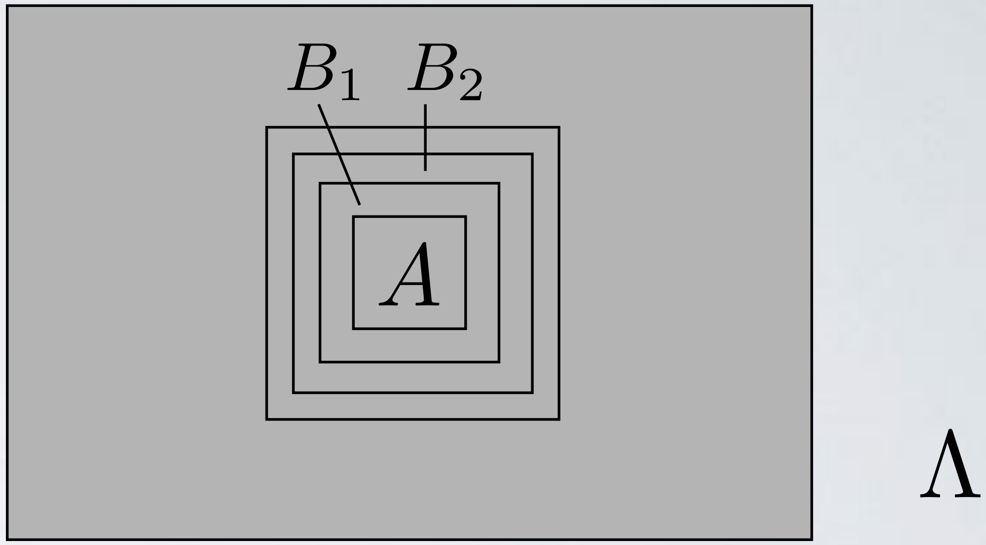
$\square$

BPT bound: $\boxed{kd^2 \leq cn}$

Proof:

Construct the largest square correctible region by adding 'onion' rings.

$\Rightarrow$ Largest square region d^2



Decompose the lattice as in Fig 2.

X and Y are correctable

$$I(X:R) = S(X) + S(R) - S(XR) = 0$$
$$S(Y) + S(R) - S(YR) = 0$$

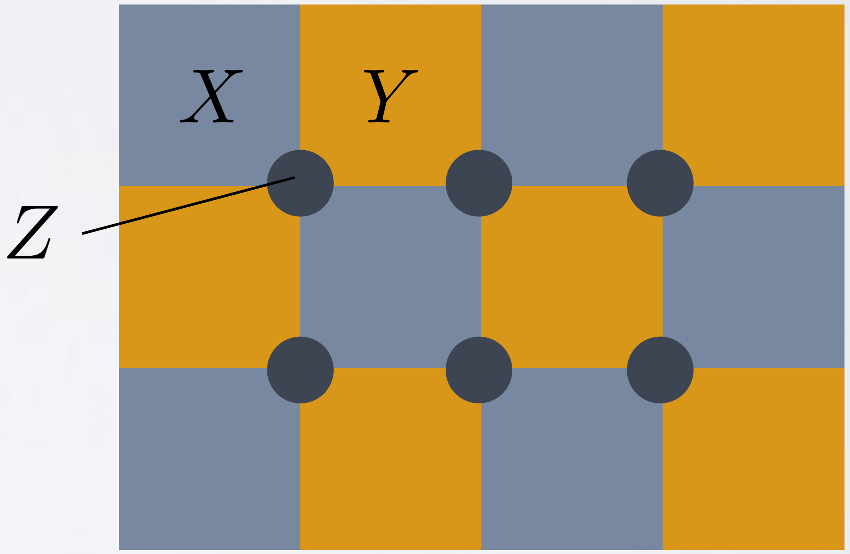


Fig 2

Sum the two and use subadditivity to get

$$S(R) \leq S(Z)$$

Take identity state on code space

$S(R) = k\log(2)$ and $S(Z) \leq cn/d^2$ $\Rightarrow$ $kd^2 \leq cn$ $\square$

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Which properties can be extended to approximate codes?

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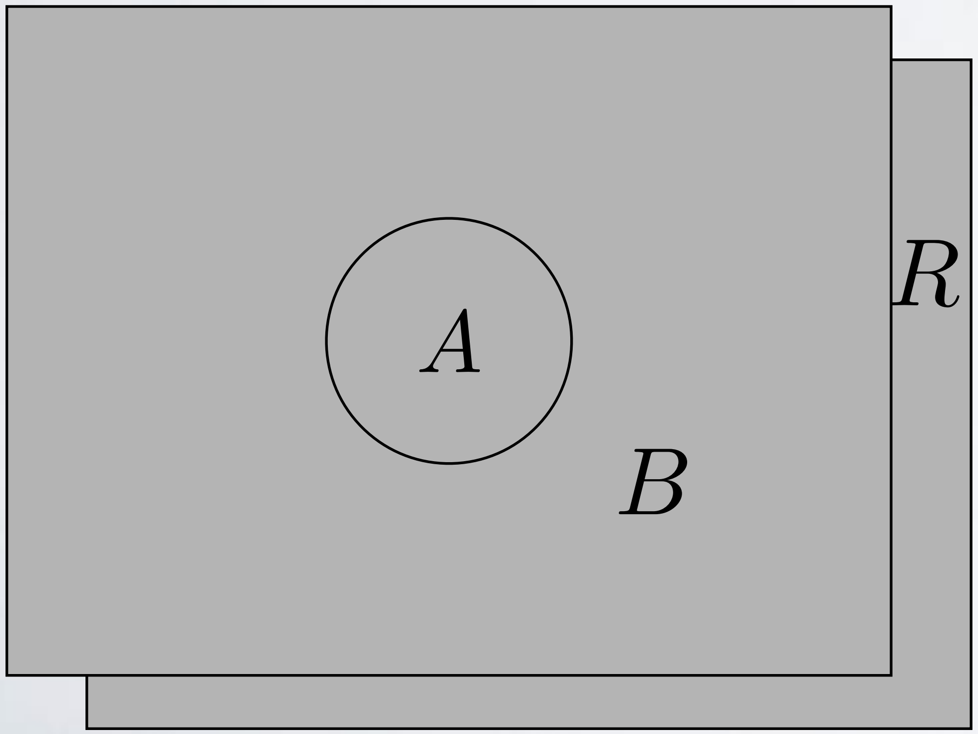
Take as our basic definition

AQEC?

Definition (approximate correctability):

There exists a recovery map $\mathcal{R}_B^{AB}$ such that for any code state $\rho^{ABR} \in \mathcal{C}$ the following holds:

$$\mathcal{B}(\rho^{ABR}, \mathcal{R}_B^{AB}(\rho^{BR})) \leq \delta$$



Bures distance $\mathcal{B}(\rho,\sigma)^2 = 1 - F(\rho,\sigma)$

$$F(\rho,\sigma) = \mathrm{tr}[\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}]$$

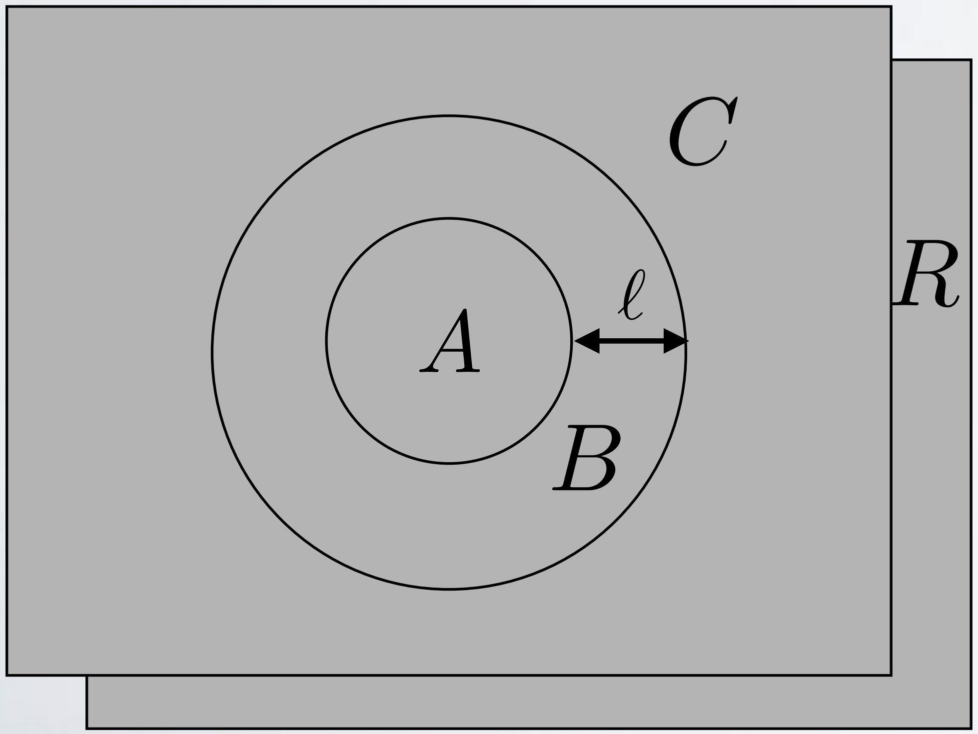
Stabilised distance; R is a copy of the logical space.

AQEC?

Definition (local approximate correctability):

There exists a recovery map $\mathcal{R}_B^{AB}$ such that for any code state $\rho^{ABCR} \in \mathcal{C}$ the following holds:

$$\mathcal{B}(\rho^{ABCR}, \mathcal{R}_B^{AB}(\rho^{BCR})) \leq \delta$$



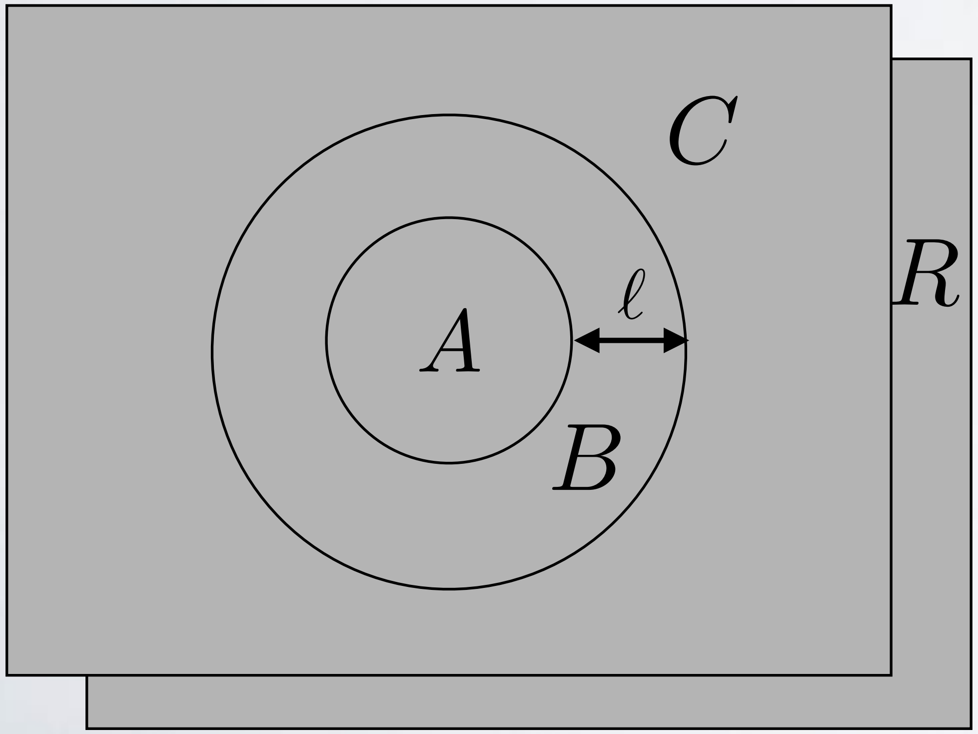
➡ state can be recovered without modifying C

EQUIVALENT FORMULATIONS

Definition (information-disturbance tradeoff):

$$\inf_{\omega^A} \sup_{\rho^{ABCR}} \mathcal{B}(\omega^A \otimes \rho^{CR}, \rho^{ACR}) = \inf_{\mathcal{R}_B^{AB}} \sup_{\rho^{ABCR}} \mathcal{B}(\mathcal{R}_B^{AB}(\rho^{BCR}, \rho^{ABCR})$$

$$\delta_\ell(A) := \inf_{\omega^A} \sup_{\rho^{ABCR}} \mathcal{B}(\omega^A \otimes \rho^{CR}, \rho^{ACR})$$



- ρ^{ABCR} is in the code space
- ω^A is some fixed state on A
- ρ^{ABCR} is in the code space

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(iii) <=> (iv)

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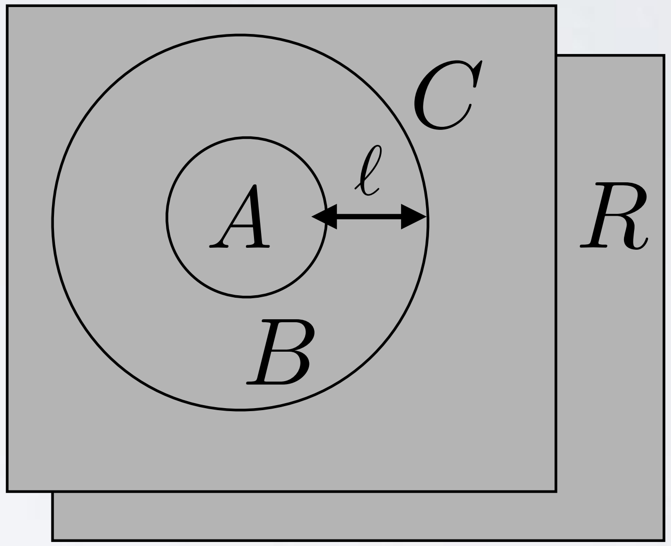

Definition (decoupling):

$$\frac{1}{9}\delta_\ell(A)^2 \leq \sup_{\rho^{ABCR}} \mathcal{B}(\rho^{ACR}, \rho^A \otimes \rho^{CR}) \leq 2\delta_\ell(A)$$

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Which properties can be extended to approximate codes?

(iii) <=> (iv)

(iii) <=> (ii) but with different error order

CLEANABILITY

Error correction $\Rightarrow$ cleanability:

If A is locally correctable: $\mathcal{B}(\mathcal{R}_B^{AB}(\rho^{BCR}), \rho^{ABCR}) \leq \delta$

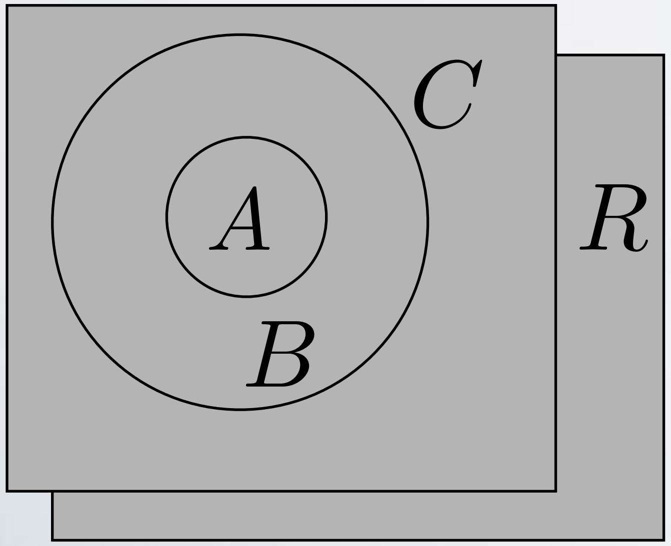
Then for any logical unitary U^{ABC}, the pull-back $V^{BC} = (\mathcal{R}_B^{AB})^*(U^{ABC})$ satisfies

$$||(U^{ABC} - V^{BC})\Pi|| \leq 4\sqrt{\delta}$$

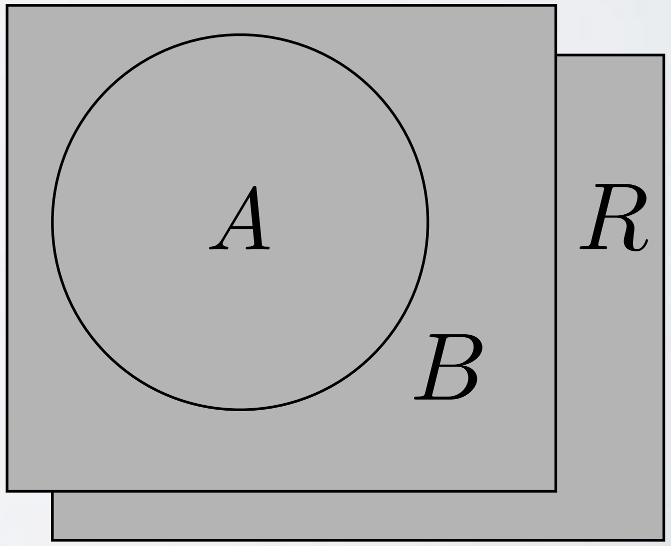
Error correction $\Leftarrow$ cleanability:

If for any U^{AB} there exists a $||V^B|| \leq 1$ on B s.t. $||(U^{ABC} - V^{BC})\Pi|| \leq \delta$

Then there exists ω^A s.t. $||\rho^{AB} - \omega^A \otimes \rho^R||_1 \leq 5\delta$



$\Rightarrow$



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(v) *Cleaning: For any unitary U preserving the code space, there exists a unitary V^{BC} such that $U|_{\mathcal{C}} = V^{BC}|_{\mathcal{C}}$*

Which properties can be extended to approximate codes?

(iii) <=> (iv)

(iii) <=> (ii) but with different error order

(iii) <=> (v) but with different error order and different locality constraints

Topological quantum order seems to be different!

APPROXIMATE BPT

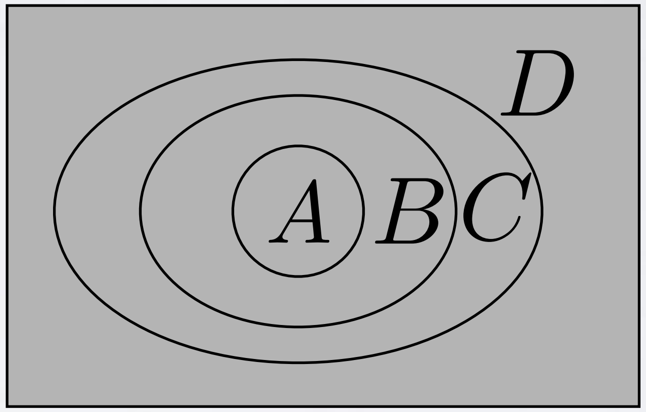
Tradeoff bound

$\boxed{kd^2 \leq cn}$ becomes $\boxed{(1 - c\frac{n\delta}{d}\log\frac{d}{n\delta})kd^2 \leq c'n\ell^4}$

Proof:

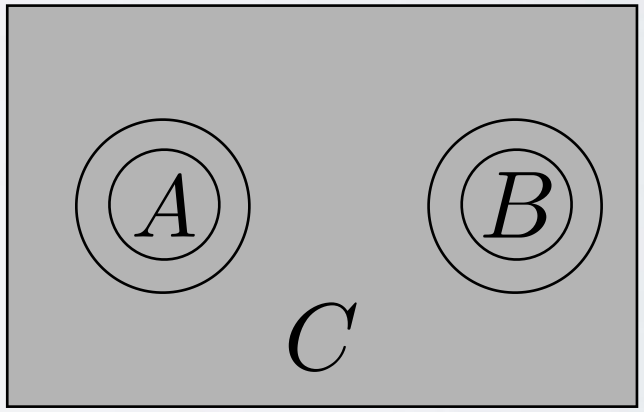
- Approximate expansion bound

  ⇨ Need (iv) and (iii)



- Approximate union bound

  ⇨ Need locality of recovery

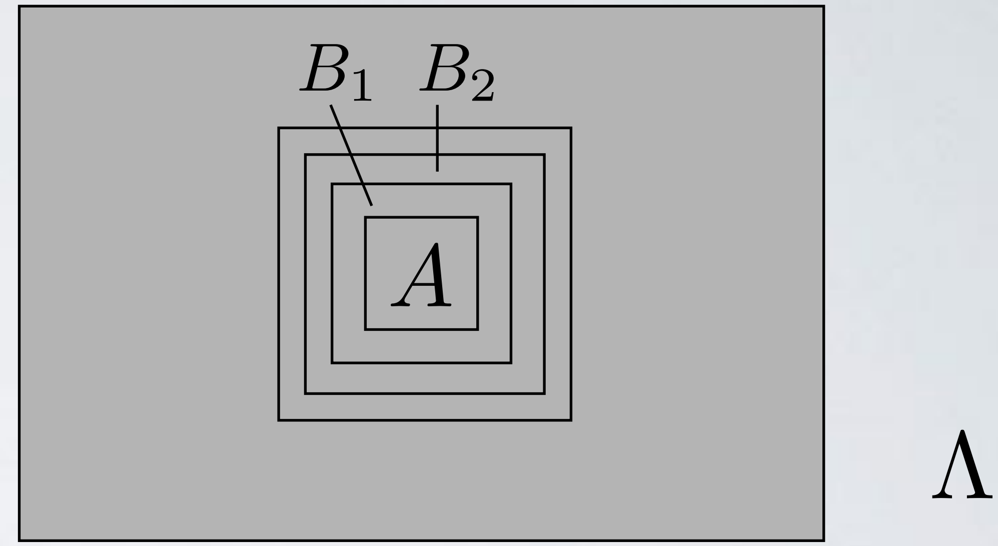


BPT bound: $$\boxed{(1 - c\frac{n\delta}{d}\log\frac{d}{n\delta})kd^2 \leq c' n \ell^4}$$

Proof:

Construct the largest square correctible region by adding 'onion' rings.

$\Rightarrow$ Largest square region d^2



Decompose the lattice as in Fig 2.

X and Y are correctable

$$I(X:R) = S(X) + S(R) - S(XR) = 0$$
$$S(Y) + S(R) - S(YR) = 0$$

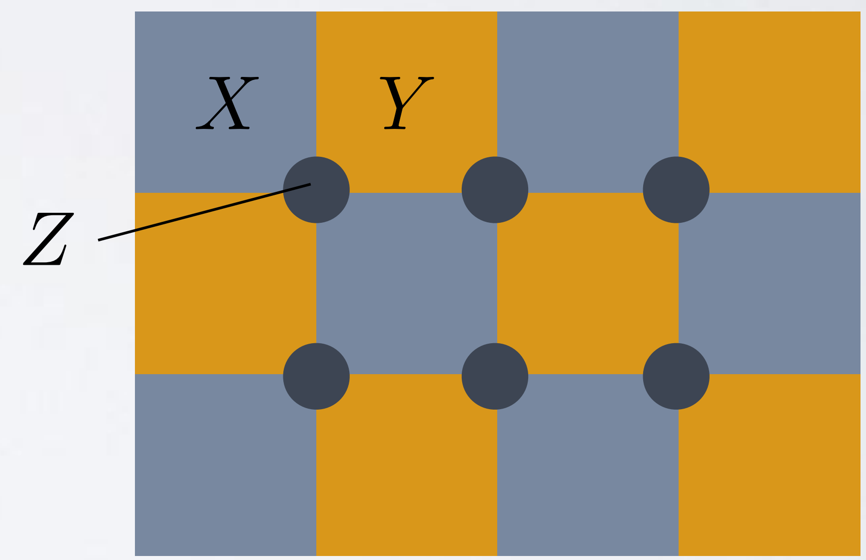


Fig 2

Sum the two and use subadditivity to get

$$S(R) \leq S(Z)$$

Take identity state on code space

$S(R) = k\log(2)$ and $S(Z) \leq cn/d^2$ $\Rightarrow$ $kd^2 \leq cn$ $\square$

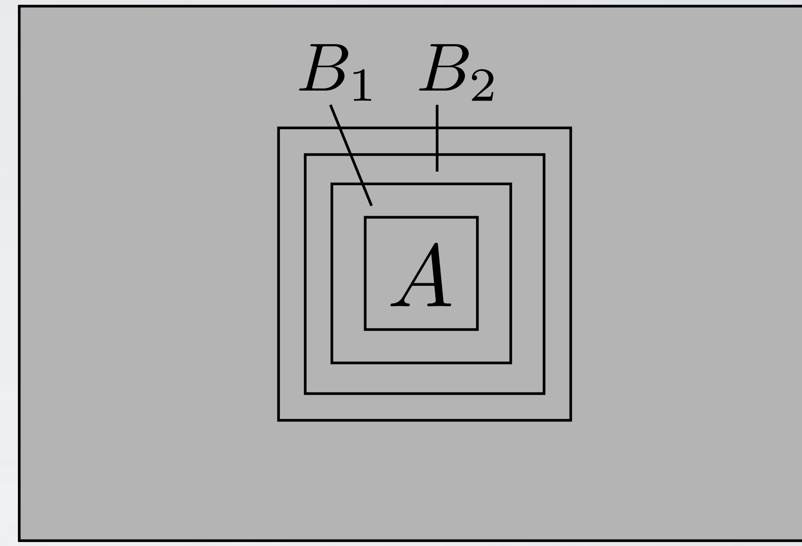
BPT bound: $(1 - c\frac{n\delta}{d}\log\frac{d}{n\delta})kd^2 \le c'n\ell^4$

Proof:

Construct the l... gion
by adding 'o...

Need (iii)=(iv)

...on d^2



Λ

Decompose the lattice as in Fig 2.

X and Y are correctable

$$I(X:R) = S(X) + S(R) - S(XR) = 0$$

$$S(Y ... R) = 0$$

Sum the tw...

Continuity of mutual information

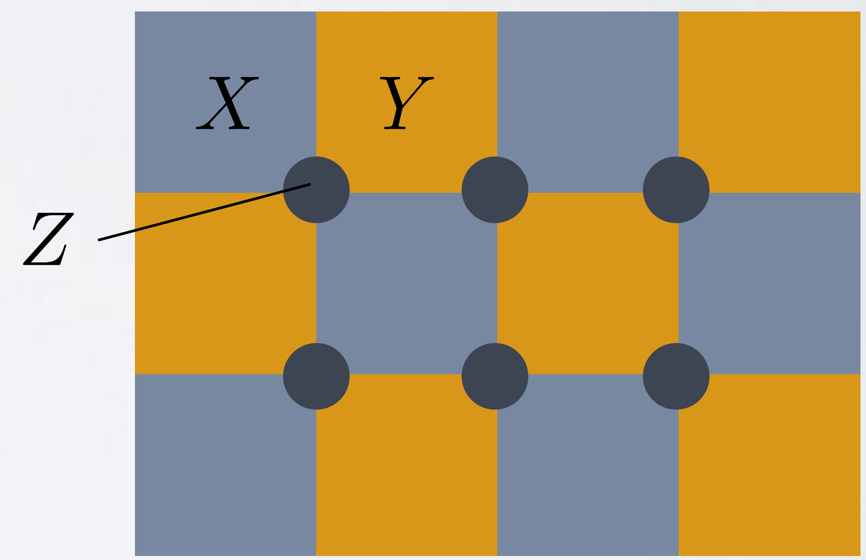


Fig 2

Take identity state on code space

$S(R) = k\log(2)$ and $S(Z) \le cn/d^2$ $\Rightarrow$ $kd^2 \le cn$ $\square$

EXAMPLES

(i) Perturbations of commuting projector codes

➡ Follows from the stability of topological order and Lieb-Robinson bounds

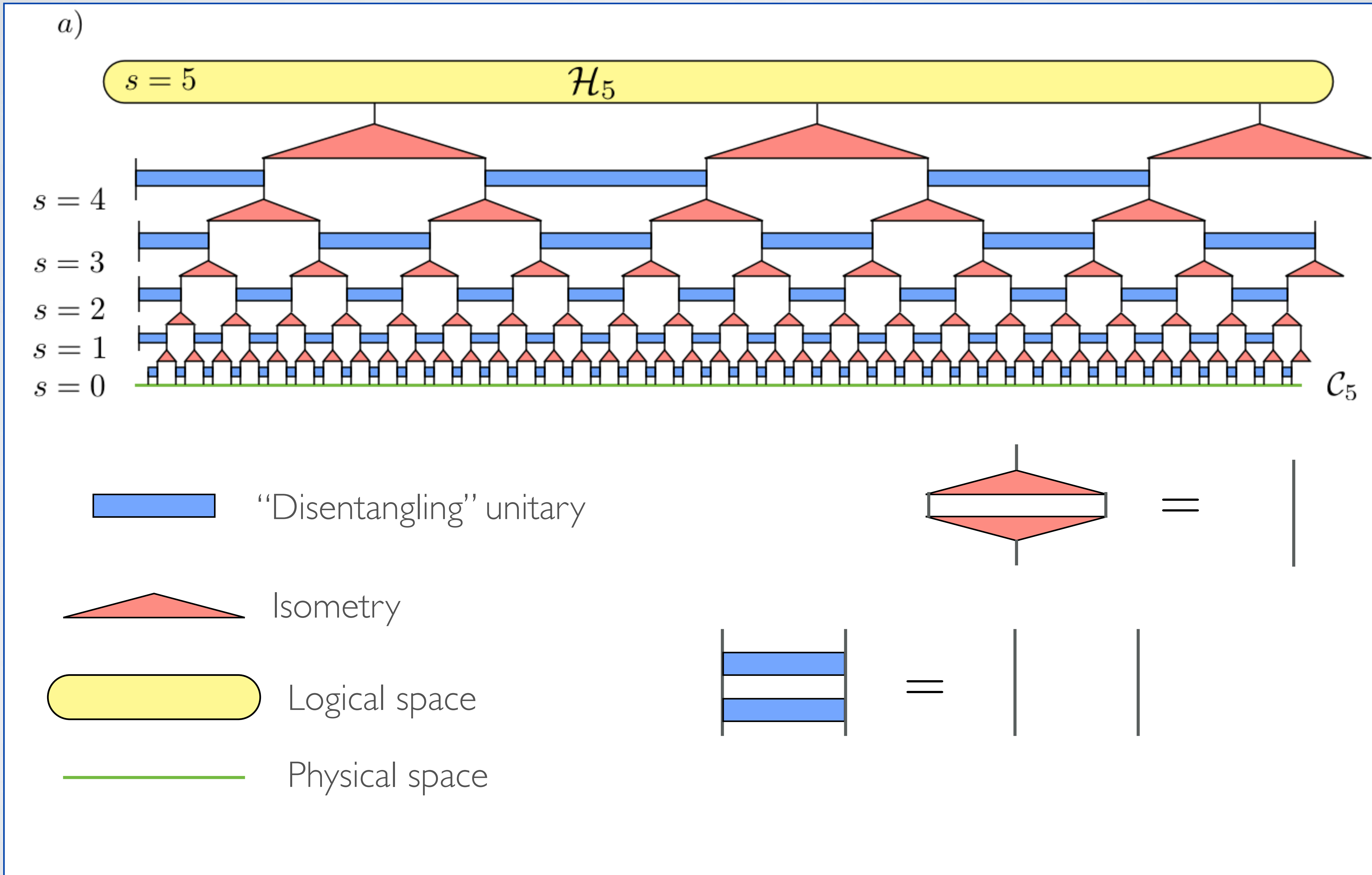
EXAMPLES

(i) Perturbations of commuting projector codes

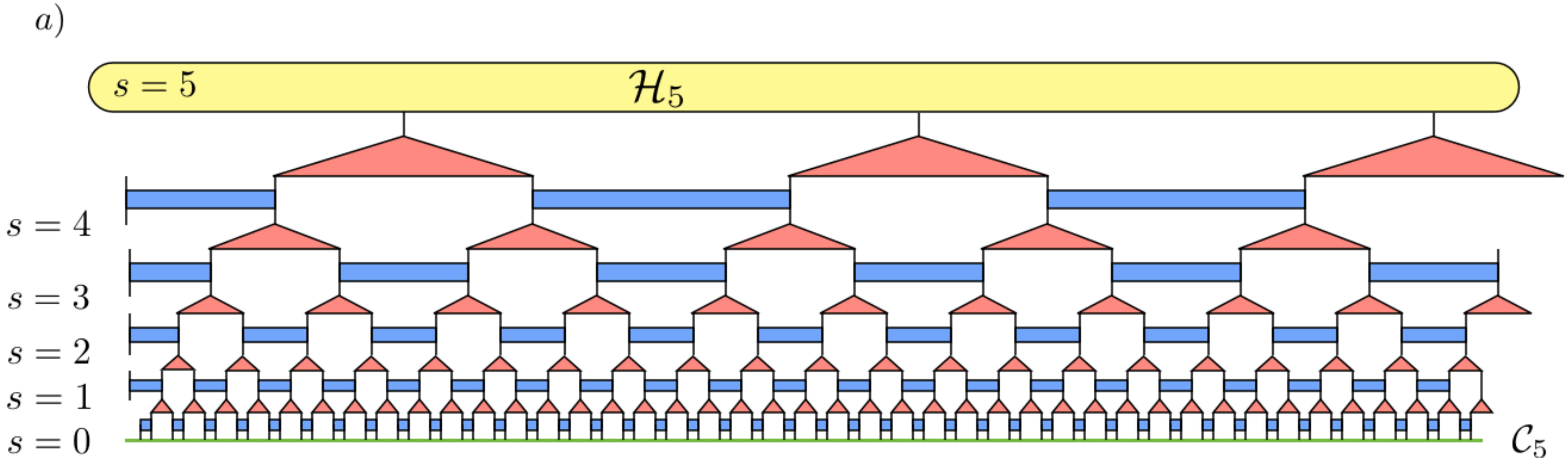
➡ Follows from the stability of topological order and Lieb-Robinson bounds

(ii) MERA codes

MERA CODES


a)
s = 5
H5
s = 4
s = 3
s = 2
s = 1
s = 0
C5
"Disentangling" unitary
Isometry
Logical space
Physical space
=
=

MERA MODEL

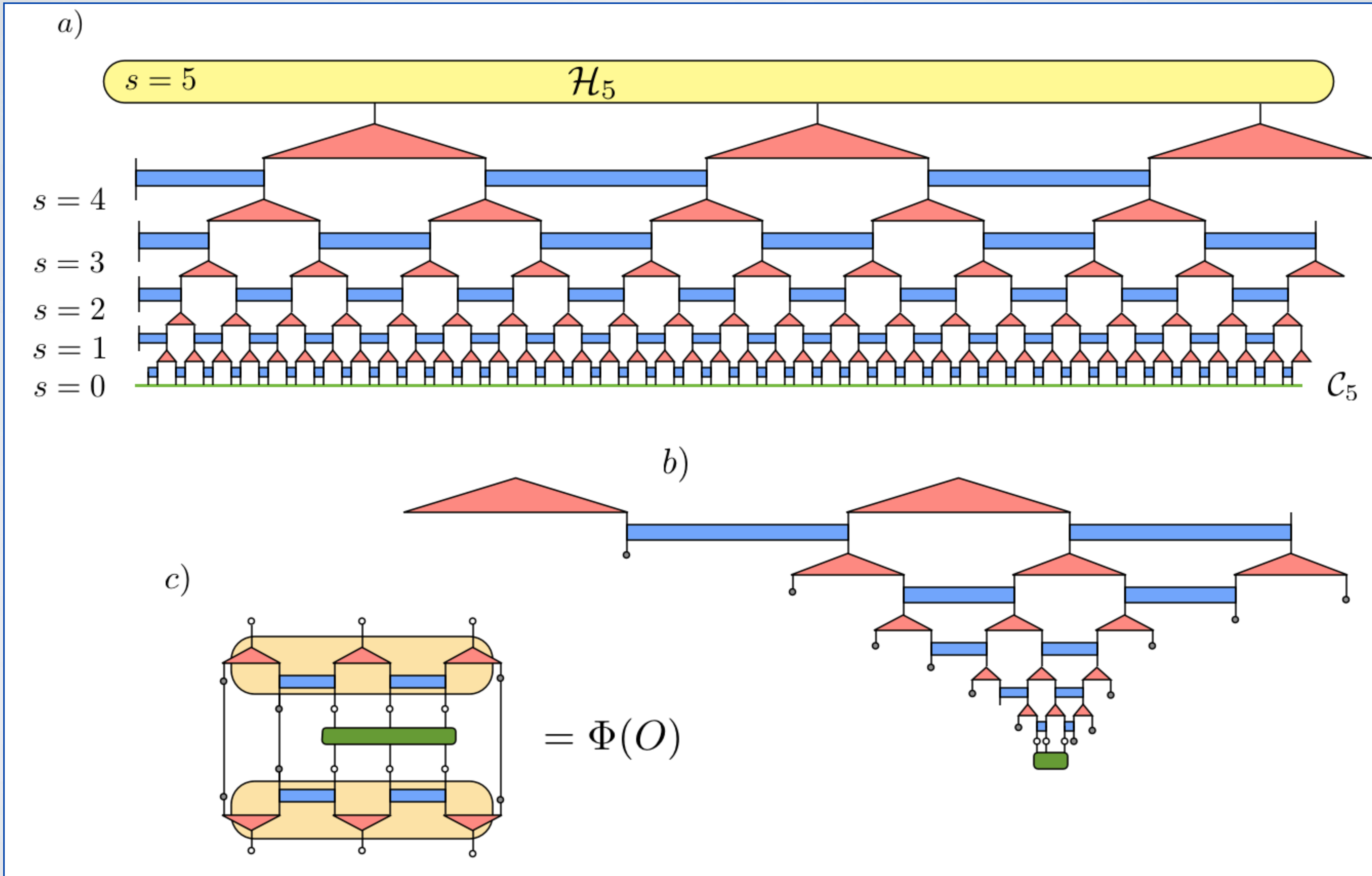


$$|\rho_s\rangle = W_1 W_2 \cdots W_s |\phi_{(s)}\rangle \qquad |\phi_{(s)}\rangle \in \mathcal{H}_s$$

The MERA circuit encodes the subspace $\mathcal{H}_s$ into $\mathcal{H}_0$ as

$$\mathcal{C}_s \subset \mathcal{H}_s$$

MERA MODEL



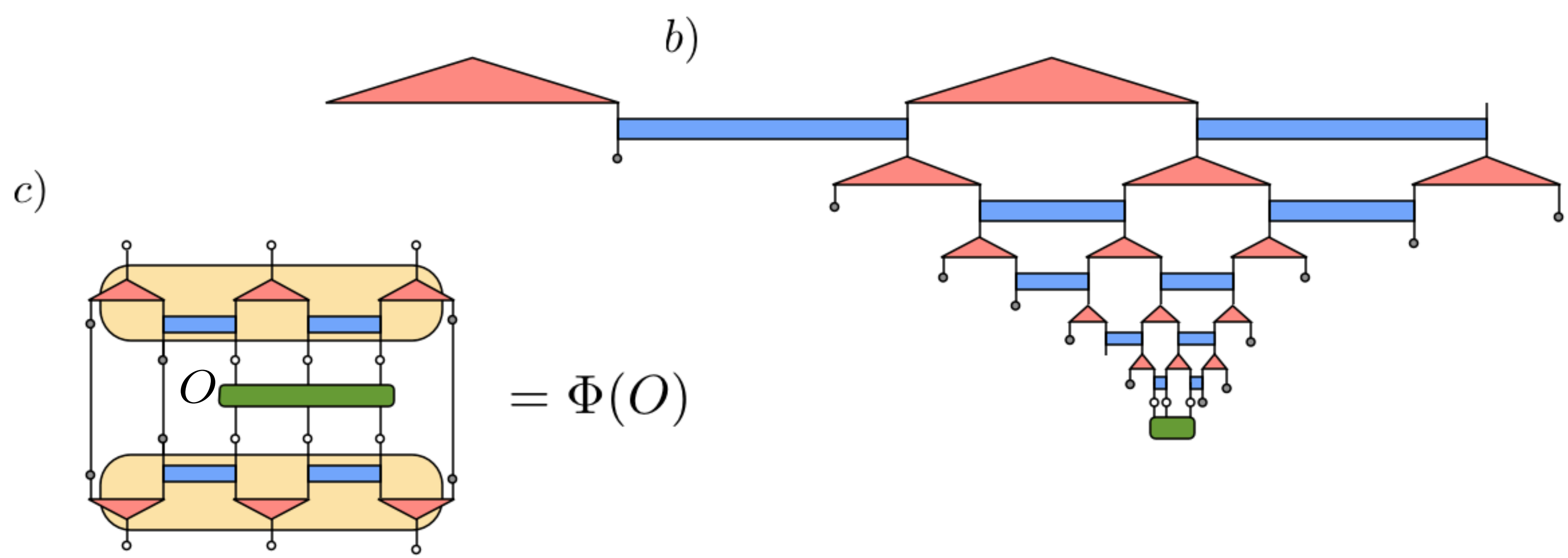
Local operators get mapped to local operators!

MERA MODEL

$$\langle \rho_s | O_s | \sigma_s \rangle = \langle \rho_{s+1} | \Phi_s^{s+1}(O_s) | \sigma_{s+1} \rangle$$

$\Phi(O)$ is a quantum channel in the Heisenberg picture

$\Phi^n(O) \approx 1\mathrm{tr}[\rho O]$ Exponentially fast in n.



Local operators get mapped to local operators!

AQEC?

Definition (information-disturbance tradeoff):

$$\inf_{\omega^A} \sup_{\rho^{ABCR}} \mathcal{B}(\omega^A \otimes \rho^{CR}, \rho^{ACR}) = \inf_{\mathcal{R}_B^{AB}} \sup_{\rho^{ABCR}} \mathcal{B}(\mathcal{R}_B^{AB}(\rho^{BCR}), \rho^{ABCR})$$

$$\frac{1}{9}\delta_\ell(A)^2 \leq \sup_{\rho^{ABCR}} \mathcal{B}(\rho^{ACR}, \rho^A \otimes \rho^{CR}) \leq 2\delta_\ell(A)$$

$$\delta_\ell(A) := \inf_{\omega^A} \sup_{\rho^{ABCR}} \mathcal{B}(\omega^A \otimes \rho^{CR}, \rho^{ACR})$$

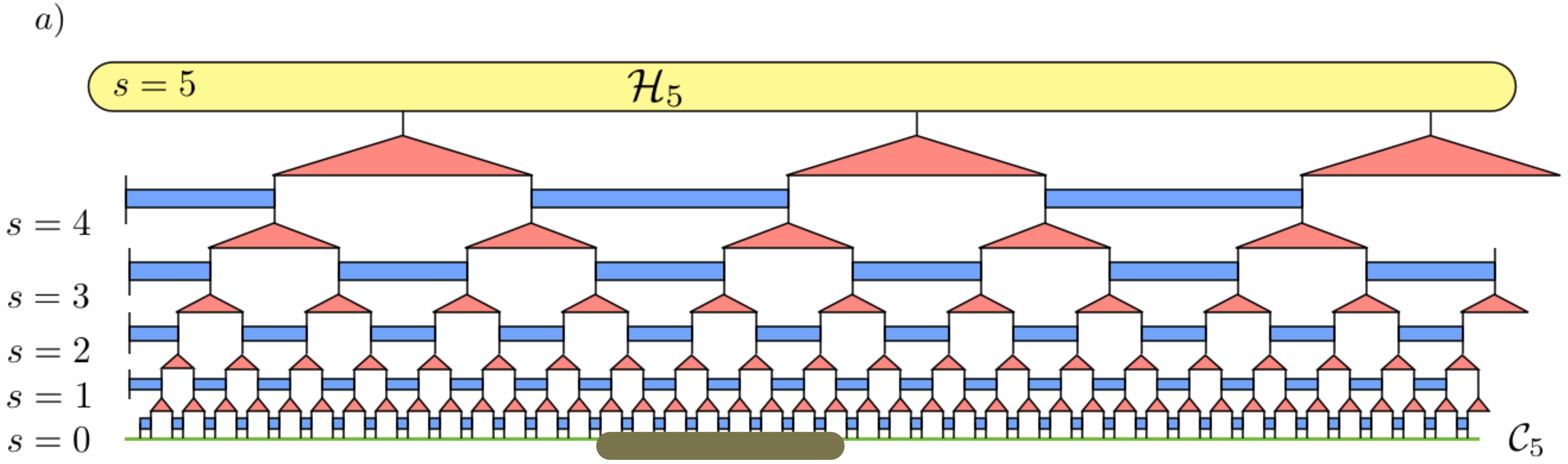
More familiar distance measure $2B^2(\rho,\sigma) \leq ||\rho - \sigma||_1 \leq 2\sqrt{2}B(\rho,\sigma)$

To show the existence of a good local recovery map, we need to bound:

$$\boxed{||\rho^A \otimes \rho^{CR} - \rho^{ACR}||_1 \quad \text{is small}}$$

Proof is very similar to showing decay of correlations

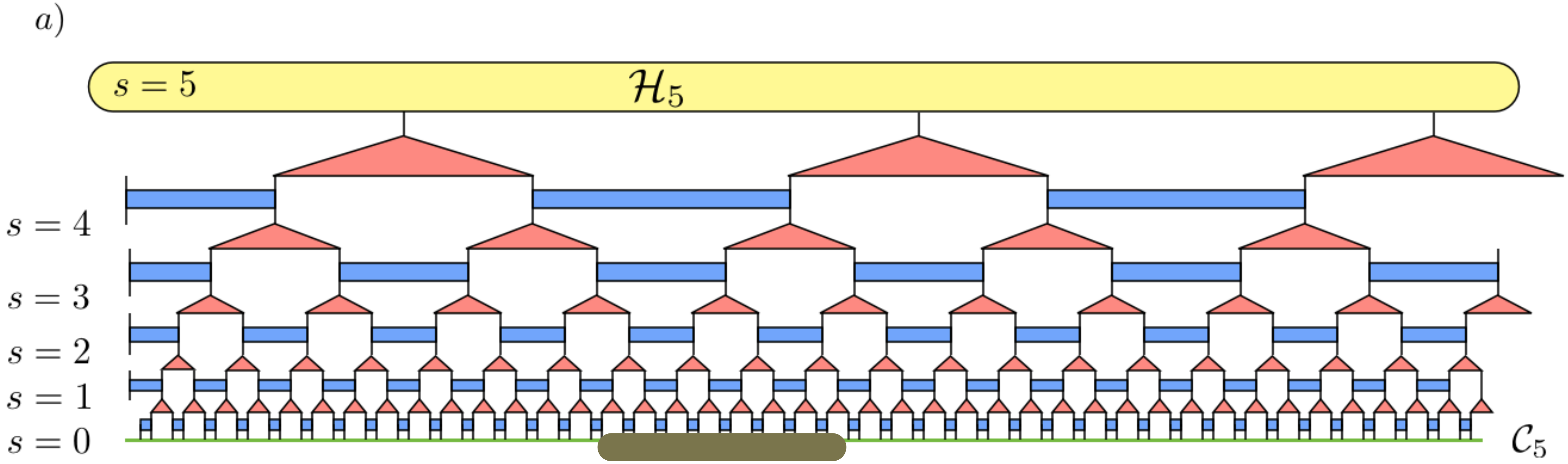
RESULT



$$||\mathcal{R}_B^{AB}(\rho^{BCR}) - \rho^{ABCR}||_1 \leq c \left(\frac{|A|}{|AB|} \right)^{\nu/2}$$

Proof is similar to that for decay of correlations in MERA

PROOF SKETCH



$$||\rho^A \otimes \rho^{CR} - \rho^{ACR}||_1 = \sup_{O_{ACR}} \mathrm{tr}[O_{ACR}(\rho^A \otimes \rho^{CR} - \rho^{ACR})]$$

$$\mathrm{tr}[O_{ACR}\rho] = \mathrm{tr}[\Phi^s(O_{ACR})\rho(s)] = \sum_j \mathrm{tr}[\Phi^s(O_{Aj}) \otimes \Phi^s(O_{CRj})\rho(s)]$$

$$\approx \sum_j \mathrm{tr}[1 \otimes \Phi^s(O_{CRj})\rho(s)]\mathrm{tr}[O_{Aj}\sigma]$$

FURTHER RESULTS

Tradeoff bound

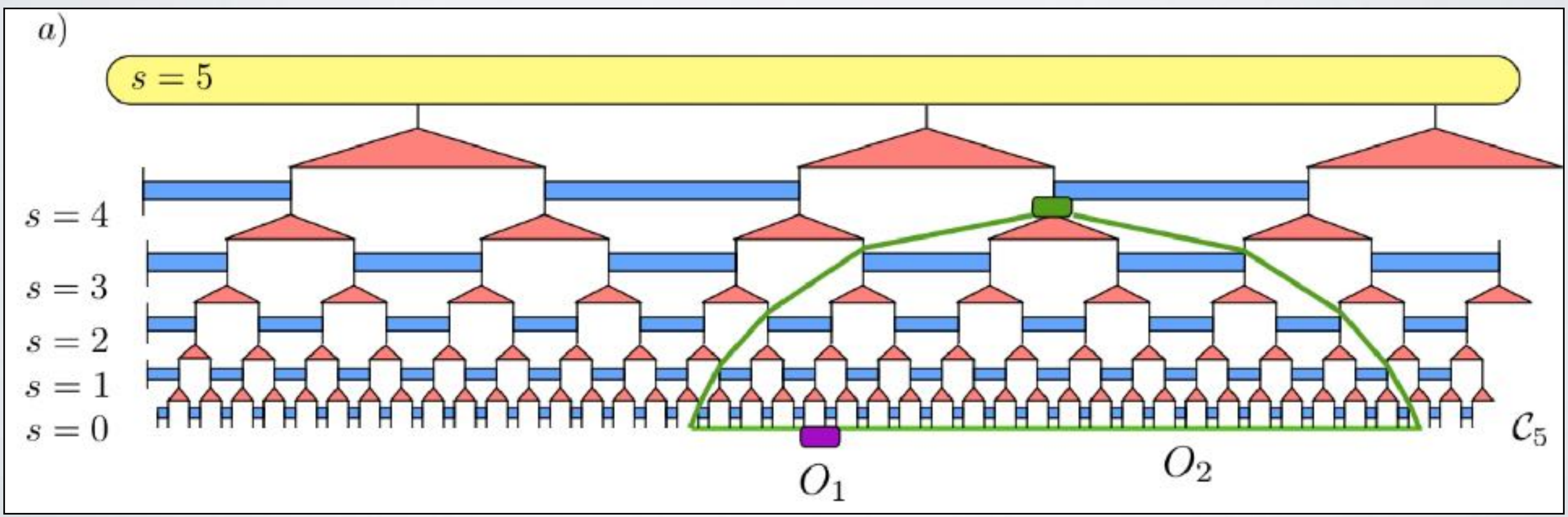
$$\boxed{K d^{\alpha} \leq c n}$$

$\alpha = 0.63$

$\alpha = 0.78$ From uberholography

Lieb-Robinson bound

$$\boxed{||[O_A, O_B(t)]|| \leq ||O_A||\ ||O_B|| e^{\log(vt) - d(A,B)/\xi}}$$



HOLOGRAPHY?

Constructive connection b/w QEC and Holography?

Useful toy model

Possible access to dynamics

Some properties not recovered (entanglement wedge hypothesis)

OPEN PROBLEMS

Further examples?

➡ Source-channel codes

Decoding MERA codes / AQEC?

Defining topological order with frustration

Dynamics or Fault tolerance?

➡ Approximate Eastin-Knill?