LOCAL APPROXIMATE ERROR CORRECTION CODES

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MOTIVATION

Intriguing example

- No quantum code can correct more than $n/4$ arbitrary errors
- Classical codes (Ex: repetition code) can correct up to $\lfloor n/2 \rfloor$ arbitrary classical errors

Crépeau et. al. (2005), quant-ph/0503139

Consequence of no-cloning theorem

Indication that approximate codes can outperform exact codes!

Crépeau et. al. (2005) construct an approximate quantum code that can correct up to $\lfloor n/2 \rfloor$ arbitrary quantum errors!
MOTIVATION

What about topological codes?

Codes often characterised by three numbers:
length \( n \); distance \( d \); encoded (qu-)bits \( k \)

Tradeoff bounds

- \( kd^2 \leq cn \)  
  Commuting projector codes
  Bravyi, Poulin, Terhal

- \( kd \leq cn \)  
  Subsystem codes
  Bravyi

- \( kd^{1/2} \leq cn \)  
  Classical lattice systems
  Bravyi, Poulin, Terhal; Yoshida

Where do approximate quantum codes sit?
Lattice commuting projector codes

\[ \{ S_j \} \quad [S_j, S_k] = 0 \quad S_j = S_j^2 \]

\[ \Pi = \prod_j S_j \quad C = \{ |\psi\rangle, \Pi |\psi\rangle = |\psi\rangle \} \]

\[ \rightarrow \quad C \text{ is the codespace} \quad \rightarrow \quad \text{Erasure errors} \]
Lattice commuting projector codes

\{ S_j \} \quad [S_j, S_k] = 0 \quad S_j = S_j^2

\Pi = \prod_j S_j \quad C = \{ |\psi\rangle, \Pi |\psi\rangle = |\psi\rangle \}

\rightarrow C \text{ is the codespace} \rightarrow \text{Erasure errors}

**Lemma**  \quad Let \( C \) be a commuting projector code, and \( ABC = \Lambda \) be decomposition of the lattice such that the distance between \( A \) and \( C \) is at least \( \ell \geq w \), the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:

(i) Topological Quantum Order (TQO): for any observable \( O_\Lambda \) with support on \( A \), any two ground states \( |\phi\rangle \) and \( |\psi\rangle \) give the same expectation value, \( \langle \phi | O^A |\phi\rangle = \langle \psi | O^A |\psi\rangle \).

(ii) Decoupling: For any \( \rho \in C \) we have \( I_\rho(A:C|R) = 0 \).

(iii) Error correction: There exists a recovery map acting on \( AB \) such that \( R_B^{AB}(\rho^{BC}) = \rho^{ABC} \) for any \( \rho \in \Pi \).

(iv) Disentangling unitary: For any \( \rho \in C \) there exists a unitary \( U^B \), such that \( U^B \rho U^B\dagger = \omega^{AB_1} \otimes \rho^{B_2C} \), for some state \( \omega^{AB_1} \).

(v) Cleaning: For any unitary \( U \) preserving the code space, there exists a unitary \( V^{BC} \) such that \( U|_C = V^{BC}|_C \).
Lemma Let $C$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between $A$ and $C$ is at least $\ell \geq w$, the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:

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(ii) Decoupling: For any $\rho \in C$ we have $I_\rho(A : CR) = 0$.

(iii) Error correction: There exists a recovery map acting on $AB$ such that $R_B^{AB}(\rho^{BC}) = \rho^{ABC}$ for any $\rho \in \Pi$.

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(i) Topological order
Lemma  Let $C$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between $A$ and $C$ is at least $\ell \geq w$, the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:

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(i) Topological order

(ii) Decoupling  $I_\rho(A : CR) = S(A) + S(AB) - S(B)$
Lemma. Let $C$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between $A$ and $C$ is at least $\ell \geq w$, the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:

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Which properties can be extended to approximate codes?

Focus on topological codes; tradeoff bounds
BPT BOUND?

Tradeoff bound

\[ kd^2 \leq cn \]

Subspace or commuting projector codes

Bravyi, Poulin, Terhal

→ Toric code saturates the bound in 2D

Proof:

- Expansion bound
- Union bound
- Counting degrees of freedom
**Expansion Lemma:**

If $A$ is correctable and $B$ is correctible, then $A \cup B$ is correctable.

**Proof:**

If $A$ is correctable

$$\rho^{ACD} = \omega^A \otimes \rho^{CD} \quad (iv)$$

If $B$ is correctable

$$\mathcal{R}_{AC}^{ABC} (\rho^{ACD}) = \rho^{ABCD} \quad (iii)$$

Define a map

$$\mathcal{F}_{C}^{ABC} (\rho^{CD}) = \mathcal{R}_{AC}^{ABC} (\omega^A \otimes \rho^{CD})$$

Show (iii)

$$\mathcal{F}_{C}^{ABC} (\rho^{CD}) = \mathcal{R}_{AC}^{ABC} (\omega^A \otimes \rho^{CD}) = \mathcal{R}_{AC}^{ABC} (\rho^{ACD}) = \rho^{ABCD}$$

$k d^2 \leq c n$
**Union Lemma:**
If $A$ is correctable and $B$ is correctible, then $A \cup B$ is correctable.

**Proof:**

\begin{align*}
A \text{ correctable } & \Rightarrow \mathcal{R}^{B \partial B}_\partial (\rho^A \setminus B) = \rho^A \tag{iv} \\
B \text{ correctable } & \Rightarrow \mathcal{R}^{B \partial B}_\partial (\rho^A \setminus A) = \rho^A \tag{iii} \\
\text{Clearly,} & \Rightarrow \mathcal{R}^{AB \partial B}_\partial (\rho^A \setminus AB) = \rho^A
\end{align*}

$k d^2 \leq cn$
Proof:

Construct the largest square correctible region by adding ‘onion’ rings.

\[
\text{Largest square region } d^2
\]

Decompose the lattice as in Fig 2.

\[
X \text{ and } Y \text{ are correctable}
\]

\[
I(X : R) = S(X) + S(R) - S(XR) = 0
\]

\[
S(Y) + S(R) - S(YR) = 0
\]

Sum the two and use subadditivity to get

\[
S(R) \leq S(Z)
\]

Take identity state on code space

\[
S(R) = k \log(2) \quad \text{and} \quad S(Z) \leq cn/d^2 \quad \Rightarrow \quad kd^2 \leq cn
\]
**Lemma**  Let $C$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between $A$ and $C$ is at least $\ell \geq w$, the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:

(i) *Topological Quantum Order (TQO):* for any observable $O_A$ with support on $A$, any two ground states $|\phi\rangle$ and $|\psi\rangle$ give the same expectation value, $\langle \phi | O^A | \phi \rangle = \langle \psi | O^A | \psi \rangle$.

(ii) *Decoupling:* For any $\rho \in C$ we have $I_\rho(A : CR) = 0$.

(iii) *Error correction:* There exists a recovery map acting on $AB$ such that $R_B^{AB}(\rho^{BC}) = \rho^{ABC}$ for any $\rho \in \Pi$.

(iv) *Disentangling unitary:* For any $\rho \in C$ there exists a unitary $U^B$, such that $U^B \rho U^B_\dagger = \omega^{AB_1} \otimes \rho^{B_2C}$, for some state $\omega^{AB_1}$.

(v) *Cleaning:* For any unitary $U$ preserving the code space, there exists a unitary $V^{BC}$ such that $U|_C = V^{BC}|_C$.

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**Which properties can be extended to approximate codes?**

Focus on topological codes; tradeoff bounds

Take as our basic definition
**AQEC?**

**Definition (approximate correctability):**

There exists a recovery map $R_B^{AB}$ such that for any code state $\rho_{ABR} \in \mathcal{C}$ the following holds:

$$\mathcal{B}(\rho_{ABR}, R_B^{AB}(\rho_{BR})) \leq \delta$$

- Bures distance: $\mathcal{B}(\rho, \sigma)^2 = 1 - F(\rho, \sigma)$
  $$F(\rho, \sigma) = \text{tr}[\sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}}]$$

- Stabilised distance; $R$ is a copy of the logical space.
AQEC?

**Definition (local approximate correctability):**

There exists a recovery map $R_{AB}^{CR}$ such that for any code state $\rho^{ABCR} \in \mathcal{C}$ the following holds:

$$\mathcal{B}(\rho^{ABCR}, R_{AB}^{CR} \rho^{BCR}) \leq \delta$$

→ state can be recovered without modifying $C$
Definition (information-disturbance tradeoff):

\[
\inf_{\omega^A} \sup_{\rho_{ABCR}} \mathcal{B}(\omega^A \otimes \rho^{CR}, \rho^{ACR}) = \inf_{\mathcal{R}_B^AB} \sup_{\rho_{ABCR}} \mathcal{B}(\mathcal{R}_B^AB (\rho^{BCR}, \rho^{ABCR})
\]

\[
\delta_\ell(A) := \inf_{\omega^A} \sup_{\rho_{ABCR}} \mathcal{B}(\omega^A \otimes \rho^{CR}, \rho^{ACR})
\]

\[\rho^{ABCR}\] is in the code space

\[\omega^A\] is some fixed state on \(A\)

\[\rho^{ABCR}\] is in the code space
Lemma  Let $C$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between $A$ and $C$ is at least $\ell \geq w$, the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:

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(iii) $\iff$ (iv)
**Definition (information-disturbance tradeoff):**

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\[
\delta_\ell(A) := \inf_{\omega^A} \sup_{\rho_{ABCR}} B(\omega^A \otimes \rho^{CR}, \rho^{ACR})
\]

**Definition (decoupling):**

\[
\frac{1}{9} \delta_\ell(A)^2 \leq \sup_{\rho_{ABCR}} B(\rho^{ACR}, \rho^A \otimes \rho^{CR}) \leq 2\delta_\ell(A)
\]
Lemma Let $C$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between $A$ and $C$ is at least $\ell \geq w$, the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:

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Which properties can be extended to approximate codes?

(iii) $\iff$ (iv)

(iii) $\iff$ (ii) but with different error order
Error correction $\Rightarrow$ cleanability:
If $A$ is locally correctable: $B(\mathcal{R}^{AB}_B(\rho^{BCR}), \rho^{ABCR}) \leq \delta$
Then for any logical unitary $U^{ABC}$, the pull-back $V^{BC} = (\mathcal{R}^{AB}_B)^*(U^{ABC})$ satisfies
$$\|(U^{ABC} - V^{BC})\Pi\| \leq 4\sqrt{\delta}$$

Error correction $\Leftarrow$ cleanability:
If for any $U^{AB}$ there exists a $\|V^B\| \leq 1$ on $B$ s.t. $\|(U^{ABC} - V^{BC})\Pi\| \leq \delta$
Then there exists $\omega^A$ s.t.
$$\|\rho^{AB} - \omega^A \otimes \rho^R\|_1 \leq 5\delta$$
Lemma  Let $C$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between $A$ and $C$ is at least $\ell \geq w$, the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:

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Which properties can be extended to approximate codes?

(iii) $\iff$ (iv)

(iii) $\iff$ (ii) but with different error order

(iii) $\iff$ (v) but with different error order and different locality constraints
APPROXIMATE BPT

Tradeoff bound

\[ kd^2 \leq cn \quad \text{becomes} \quad (1 - c \frac{n\delta}{d} \log \frac{d}{n\delta})kd^2 \leq c'n\ell^4 \]

Proof:

- Approximate expansion bound
- Need (iv) and (iii)

- Approximate union bound
- Need locality of recovery
Proof:

Construct the largest square correctible region by adding ‘onion’ rings.

Decompose the lattice as in Fig 2.

$X$ and $Y$ are correctable

$I(X : R) = S(X) + S(R) - S(XR) = 0$

$S(Y) + S(R) - S(YR) = 0$

Sum the two and use subadditivity to get

$S(R) \leq S(Z)$

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\[\square\]
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Need (iii) = (iv)

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\[ S(R) \leq cn/d^2 \quad \Rightarrow \quad kd^2 \leq cn \]
EXAMPLES

(i) Perturbations of commuting projector codes

Follows from the stability of topological order and Lieb-Robinson bounds
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(i) Perturbations of commuting projector codes

Follows from the stability of topological order and Lieb-Robinson bounds

(ii) MERA codes
MERA CODES

\[ s = 5 \]
\[ \mathcal{H}_5 \]

\[ s = 4 \]
\[ s = 3 \]
\[ s = 2 \]
\[ s = 1 \]
\[ s = 0 \]

“Disentangling” unitary

Isometry

Logical space

Physical space
The MERA circuit encodes the subspace $\mathcal{H}_s$ into $\mathcal{H}_0$ as

$$|\rho_s\rangle = W_1 W_2 \cdots W_s |\phi(s)\rangle \quad |\phi(s)\rangle \in \mathcal{H}_s$$

$C_s \subset \mathcal{H}_s$
Local operators get mapped to local operators!
\[ \langle \rho_s | O_s | \sigma_s \rangle = \langle \rho_{s+1} | \Phi_{s+1}^s (O_s) | \sigma_{s+1} \rangle \]

\( \Phi(O) \) is a quantum channel in the Heisenberg picture

\( \Phi^n(O) \approx 1 \text{tr}[\rho O] \) Exponentially fast in n.

Local operators get mapped to local operators!
Definition (information-disturbance tradeoff):

$$\inf_{\omega^A} \sup_{\rho_{ABCR}} \mathcal{B}(\omega^A \otimes \rho^{CR}, \rho^{ACR}) = \inf_{\mathcal{R}_B^{AB}} \sup_{\rho_{ABCR}} \mathcal{B}(\mathcal{R}_B^{AB}(\rho^{BCR}), \rho^{ABCR})$$

$$\frac{1}{9} \delta_\ell(A)^2 \leq \sup_{\rho_{ABCR}} \mathcal{B}(\rho^{ACR}, \rho^A \otimes \rho^{CR}) \leq 2\delta_\ell(A)$$

$$\delta_\ell(A) := \inf_{\omega^A} \sup_{\rho_{ABCR}} \mathcal{B}(\omega^A \otimes \rho^{CR}, \rho^{ACR})$$

More familiar distance measure

$$2B^2(\rho, \sigma) \leq \|\rho - \sigma\|_1 \leq 2\sqrt{2}B(\rho, \sigma)$$

To show the existence of a good local recovery map, we need to bound:

$$\|\rho^A \otimes \rho^{CR} - \rho^{ACR}\|_1$$ is small

Proof is very similar to showing decay of correlations
Disentangling unitary Isometry

Logical space

Physical space

\[\|R_{AB}^B (\rho^{BCR}) - \rho^{ABCR}\|_1 \leq c \left( \frac{|A|}{|AB|} \right)^{\nu/2}\]

Proof is similar to that for decay of correlations in MERA
PROOF SKETCH

\[ \begin{align*}
\|\rho^A \otimes \rho^{CR} - \rho^{ACR}\|_1 &= \sup_{O_{ACR}} \text{tr}[O_{ACR}(\rho^A \otimes \rho^{CR} - \rho^{ACR})] \\
\text{tr}[O_{ACR}\rho] &= \text{tr}[\Phi^s(O_{ACR})\rho(s)] = \sum_j \text{tr}[\Phi^s(O_{Aj}) \otimes \Phi^s(O_{CRj})\rho(s)] \\
&\approx \sum_j \text{tr}[1 \otimes \Phi^s(O_{CRj})\rho(s)]\text{tr}[O_{Aj}\sigma]
\end{align*} \]
FURTHER RESULTS

Tradeoff bound

\[ K d^\alpha \leq cn \]
\[ \alpha = 0.63 \]
\[ \alpha = 0.78 \] From uberholography

Lieb-Robinson bound

\[ \| [O_A, O_B(t)] \| \leq \| O_A \| \| O_B \| e^{\log(vt) - d(A,B)/\xi} \]
HOLOGRAPHY?

Constructive connection b/w QEC and Holography?

Useful toy model

Possible access to dynamics

Some properties not recovered (entanglement wedge hypothesis)
OPEN PROBLEMS

Further examples?

Source-channel codes

Decoding MERA codes / AQEC?

Defining topological order with frustration

Dynamics or Fault tolerance?

Approximate Eastin-Knill?