LOCAL RECOVERY MAPS AS DUCT TAPE FOR MANY BODY SYSTEMS

Michael J. Kastoryano

November 14 2016, QuSoft Amsterdam
CONTENTS

Local recovery maps

Exact recovery and approximate recovery

Local recovery for many body systems

Hammersley-Clifford and Gibbs sampling

State preparation

Evaluating local expectation values

Efficient state preparation

Further Applications
LOCAL RECOVERY MAPS

Strong subadditivity (SSA):

\[ I_\rho(A : C|B) = S(AB) + S(BC) - S(B) - S(ABC) \geq 0 \]

Equality

\[ I_\rho(A : C|B) = 0 \iff R_{AB}(\rho_{BC}) = \rho \]

Petz map

\[ R_{AB}(\sigma) = \rho_1^{1/2} \rho_B^{-1/2} \sigma^{1/2} \rho_B^{-1/2} \rho_1^{1/2} \]


Markov State

\[ \rho = \bigoplus_j \rho_{AB}^L_j \otimes \rho_{B^R}^j \]

P. Hayden, et. al., CMP 246 (2004)

there exists a disentangling unitary on B.
LOCAL RECOVERY MAPS

Strengthening SSA:

\[ I_\rho(A : C|B) \geq -2 \log_2 F(\rho, R_{AB}(\rho_{AB})) \]

Rotated Petz map

\[ R_{AB}(\sigma) = \int dt \beta(t) \rho^{1/2+it}_{AB} \rho^{-1/2-it}_B \sigma \rho^{-1/2+it}_B \rho^{1/2-it}_{AB} \]

ABC are arbitrary

Is the map universal?
Is the conditional mutual information necessary?
Other properties of the map?

APPLICATIONS

Shannon Theory and Entanglement theory
APPLICATIONS

Shannon Theory and Entanglement theory

Classical Simulations
Tensor networks, stoquastic models

Quantum Simulations (sampling)
Topological order

Quantum error correction

Renormalization Group, critical models, AdS/CFT
APPLICATIONS

Shannon Theory and Entanglement theory

Classical Simulations

Tensor networks, stoquastic models

Quantum Simulations (sampling)

Topological order

Quantum error correction

Renormalization Group, critical models, AdS/CFT
MANY-BODY SETTING

Exact recovery

For any A, and B shielding A:

\[ I_\rho(A : C|B) = 0 \]

\[ \mathcal{H} = \mathcal{H}_{2^N} \]
HAMMERSLEY-CLIFFORD

Exact recovery

For any $A$, and $B$ shielding $A$:
\[ I_\rho(A : C|B) = 0 \]

\[ \rho > 0 \text{ is the Gibbs state of a local commuting } H \]

\[ \rho = |\psi\rangle\langle\psi| \text{ is the ground state of a local commuting } H \]

\[ \mathcal{H} = \mathcal{H}_2^\otimes N \]

HAMMERSLEY-CLIFFORD

Exact recovery

For any $A$, and $B$ shielding $A$:

$$I_\rho(A : C|B) = 0$$

$\rho > 0$ is the Gibbs state of a local commuting $H$

$\rho = |\psi\rangle\langle\psi|$ is the ground state of a local commuting $H$

Approximate recovery

For any $A$, and $B$ shielding $A$:

$$I_\rho(A : C|B) \leq Ke^{-c\ell}$$

$\mathcal{H} = \mathcal{H}_2^\otimes N$

\[C\]
\[
\ell
\]
\[A\]

\[B\]

HAMMERSLEY-CLIFFORD

Exact recovery

For any A, and B shielding A:
\[ I_\rho(A : C | B) = 0 \]

\( \rho > 0 \) is the Gibbs state of a local commuting \( H \)

\( \rho = |\psi\rangle\langle\psi| \) is the ground state of a local commuting \( H \)


Approximate recovery

For any A, and B shielding A:
\[ I_\rho(A : C | B) \leq Ke^{-c\ell} \]

\( \rho > 0 \) is the Gibbs state of a local non-commuting \( H \)

\( \rho = |\psi\rangle\langle\psi| \) is the ground state of a gaped local non-commuting \( H \)

Further consequences

\[ I(A : B_1 \cdots B_{n+1}) - I(A : B_1 \cdots B_n) = I(A : B_{n+1} | B_1 \cdots B_n) \]

Mutual info area law: \[ I(A : A^c) \leq c|\partial A| \]

Decaying CMI provides a quantitative MI area law
Further consequences

\[ I(A : B_1 \cdots B_{n+1}) - I(A : B_1 \cdots B_n) = I(A : B_{n+1} | B_1 \cdots B_n) \]

**Mutual info area law:** \( I(A : A^c) \leq c |\partial A| \)

Decaying CMI provides a quantitative MI area law

Can also show: Small CMI implies efficient MPS/MPO representation!

**Take-home message:** CMI replaces Area Law, HC program replaces the area law conjecture
Further consequences

\[ I(A : B_1 \cdots B_{n+1}) - I(A : B_1 \cdots B_n) = I(A : B_{n+1}|B_1 \cdots B_n) \]

Mutual info area law: \( I(A : A^c) \leq c|\partial A| \)

Decaying CMI provides a quantitative...

Can also show:
Small CMI implies efficient MPS/MPO representation!

What about dynamics and state preparation?

Take-home message: CMI replaces Area Law, HC program replaces the area law conjecture
MONTE-CARLO SIMULATIONS

Want to evaluate: $\langle Q \rangle = \sum_x \pi(x) Q(x)$

$Idea$: - obtain a sample configuration from the distribution $\pi$
- Set up a Markov chain with $\pi$ as an approximate fixed point

$\pi \propto e^{-\beta H}$

classical Gibbs state
MONTE-CARLO SIMULATIONS

Want to evaluate: \[ \langle Q \rangle = \sum_x \pi(x) Q(x) \]

\( \pi \propto e^{-\beta H} \)

classical Gibbs state

Idea:
- obtain a sample configuration from the distribution \( \pi \)
- Set up a Markov chain with \( \pi \) as an approximate fixed point

Metropolis algorithm:
- start with random configuration
- Flip a spin at random, calculate energy
- If energy decreased, accept the flip
- If energy increased, accept the flip with probability \( p_{\text{flip}} = e^{-\beta \Delta E} \)
- Repeat until equilibrium is reached
MONTE-CARLO SIMULATIONS

Want to evaluate:  
\[ \langle Q \rangle = \sum_x \pi(x) Q(x) \]

\[ \pi \propto e^{-\beta H} \]

Classical Gibbs state

Idea:  
- Obtain a sample configuration from the distribution \( \pi \)
- Set up a Markov chain with \( \pi \) as an approximate fixed point

Metropolis algorithm:  
- Start with random configuration
- Flip a spin at random, calculate energy
- If energy decreased, accept the flip
- If energy increased, accept the flip with probability \( p_{\text{flip}} = e^{-\beta \Delta E} \)
- Repeat until equilibrium is reached

Equilibrium?
ANALYTIC RESULTS

Note:  - Glauber dynamics (Metropolis) is modeled by a semigroup

\[ P_t = e^{tL} \]
ANALYTIC RESULTS

Note: - Glauber dynamics (Metropolis) is modeled by a semigroup

\[ P_t = e^{tL} \]

Fundamental result for Glauber dynamics:

- \( \pi \) has exponentially decaying correlations
- \( P_t \) mixes in time \( O(\log(N)) \)
- \( L \) is gapped

Independent of boundary conditions in 2D
Independent of specifics of the model
No intermediate mixing

F. Martinelli, Lect. Prof. Theor. Stats, Springer
A. Guionnet, B. Zegarlinski, Sem. Prob., Springer
Davies maps are another generalization of Glauber dynamics.

\[ T_t = e^{t\mathcal{L}} \]

\[ \mathcal{L} = \sum_{j \in \Lambda} (R_j \partial - \text{id}) \]

\( R_j \partial \) is the Petz recovery map!

MJK and K. Temme, arXiv:1505.07811
QUANTUM GIBBS SAMPLERS

Commuting Hamiltonian

Davies maps are another generalization of Glauber dynamics

\[ T_t = e^{t\mathcal{L}} \]
\[ \mathcal{L} = \sum_{j \in \Lambda} (R_j \partial - id) \]

\( R_j \partial \) is the Petz recovery map!

The exists a partial extension of the **statics = dynamics** theorem

MJK and K. Temme, arXiv:1505.07811

MJK and F. Brandao, CMP 344 (2016)
QUANTUM GIBBS SAMPLERS

Commuting Hamiltonian

Davies maps are another generalization of Glauber dynamics

\[ T_t = e^{t \mathcal{L}} \]
\[ \mathcal{L} = \sum_{j \in \Lambda} (R_j \partial - id) \]
\[ R_j \partial \] is the Petz recovery map!

The exists a partial extension of the statics = dynamics theorem

Non-commuting Hamiltonian

\[ \mathcal{L} = \sum_{j \in \Lambda} (R_j \partial - id) \]
\[ R_j \partial \] is the rotated Petz map!

no longer frustration-free
Theorem \( \star \) does not hold
Davies maps are non-local

MJK and K. Temme, arXiv:1505.07811
MJK and F. Brandao, CMP 344 (2016)
STATE PREPARATION

Based on: MJK, F. Brandao, arXiv:1609.07877
**Setting**

**Lattice:**

\[ A \subset \Lambda \]

**Hamiltonian:**

\[ H_A = \sum_{Z \subset A} h_Z \]

\[ h_Z = 0 \text{ for } |Z| \geq K \]

**Gibbs states:**

\[ \rho^A = e^{-\beta H_A} / \text{Tr}[e^{-\beta H_A}] \]

is the Gibbs state restricted to \( A \)

**Note:** Superscript for domain of definition of Gibbs state, while subscript for partial trace.
**THE MARKOV CONDITION**

**Uniform Markov:**

Any subset \( X = ABC \subset \Lambda \) with \( B \) shielding \( A \) from \( C \) in \( X \), we have

\[
I_{\rho^X} (A : C|B) \leq \delta(\ell)
\]

Recall:

\[
\rho^X = e^{-\beta H_X} / \text{Tr}[e^{-\beta H_X}]
\]

Also must hold for non-contractible regions
CORRELATIONS

Uniform Clustering:

Any subset $X = ABC \subset \Lambda$ with $\text{supp}(f) \subset A$ and $\text{supp}(g) \subset B$

$$\text{Cov}_\rho X(f, g) \leq \epsilon(\ell)$$

$\text{Cov}_\rho (f, g) = |\text{tr}[\rho fg] - \text{tr}[\rho f] \text{tr}[\rho g]|$

Note: Uniform Clustering follows from uniform Gap
LOCAL PERTURBATIONS

Commuting Hamiltonian

\[ e^{-\beta(H^A + H^B)} = e^{-\beta H^A} e^{-\beta H^B} \]

if \[ [H^A, H^B] = 0 \]

Non-commuting Hamiltonian

General

\[ e^{-\beta(H+V)} = O_V e^{-\beta H} O_V^\dagger \]

\[ \|O_V - O_V^\ell\| \leq c_1 e^{-c_2 \ell} \equiv \gamma(\ell) \]

\[ \|O_V\| \leq e^{\beta \|V\|} \]

Only works if \( V \) is local!
**Uniform Markov**

\[ I_{\rho x} (A : C | B) \leq \delta(\ell) \]

**Uniform clustering**

\[ \text{Cov}_{\rho x} (f, g) \leq \epsilon(\ell) \]

**Local perturbations**

\[ \| e^{-\beta(H+V)} - O_V^\ell e^{-\beta H} O_V^\ell \| \leq c_1 e^{-c_2 \ell} \equiv \gamma(\ell) \]
**Result 1:**

Any subset $X = ABC \subset \Lambda$ with $B$ shielding $A$ from $C$ in $X$, if $\rho$ is uniformly clustering,

$$\|\text{tr}_{BC}[\rho^{ABC}] - \text{tr}_{B}[\rho^{AB}]\|_1 \leq c|AB|(\epsilon(\ell) + \gamma(\ell))$$

**Consequence:** Efficient evaluation of local expectation values

$$\langle O_A \rangle = \text{tr}[\rho^\Lambda O_A] \approx \text{tr}[\rho^{AB} O_A]$$
**Result 1:**

Any subset $X = ABC \subset \Lambda$ with $B$ shielding $A$ from $C$ in $X$, if $\rho$ is uniformly clustering,

$$\| \text{tr}_{BC}[\rho^{ABC}] - \text{tr}_B[\rho^{AB}] \|_1 \leq c|AB|(\epsilon(\ell) + \gamma(\ell))$$

Proof idea:

Remove pieces of the boundary of $B$ one by one

Telescopic sum

$$\| \text{tr}_{BC}[\rho^X - \rho^{AB} \otimes \rho^C] \|_1 \leq \sum_j \| \text{tr}_{BC}[\rho^{X_{j+1}} - \rho^{X_j}] \|_1$$

Bound each term

$$\| \text{tr}_{BC}[\rho^{X_{j+1}} - \rho^{X_j}] \|_1 \approx \sup_{g_A} \| \text{tr}[g_A(O_j^\ell \rho_{X_j} O_j^{\ell,\dagger} - \rho^{X_j})] \|_1$$

$$= \text{Cov}_{\rho^{X_j}}(g_A, O_j^{\ell,\dagger} O_j^\ell)$$
Main Result:

If $\rho$ is uniformly clustering and uniformly Markov, then there exists a depth $D + 1$ circuit of quantum channels $F = F_{D+1} \cdots F_1$ of local range $O(\log(L))$, such that

$$||F(\psi) - \rho||_1 \leq cL^D(\epsilon(\ell) + \delta(\ell) + \gamma(\ell))$$

MJK, F. Brandao, arXiv:1609.07877
Main Result:

If $\rho$ is uniformly clustering and uniformly Markov, then there exists a depth $D + 1$ circuit of quantum channels $F = F_{D+1} \cdots F_1$ of local range $O(\log(L))$, such that

$$||F(\psi) - \rho||_1 \leq cL^D(\epsilon(\ell) + \delta(\ell) + \gamma(\ell))$$

Corollary:

If $\rho$ is uniformly clustering and uniformly Markov, then there exists a depth $M = O(\log(L))$ circuit of strictly local quantum channels $F = F_M \cdots F_1$, such that

$$||F(\psi) - \rho||_1 \leq cL^D(\epsilon(\ell) + \delta(\ell) + \gamma(\ell))$$
Step 1:

- Cover the lattice in concentric squares $A_- \subset A \subset A_+$
- By the Markov condition
  $$\|R_{A_+}^{\rho} (\rho_{A^c}) - \rho\|_1 \leq N_A (\gamma(\ell) + \delta(\ell))$$
- By Local indistinguishability
  $$\|\text{tr}_A [\rho_{A^c}] - \rho_{A^c}\|_1 \leq N_A \epsilon(\ell)$$

Local cpt map $F_A \equiv R_{A_+}^{\rho} \text{tr}_A$

$$\|F_A (\rho_{A^c}) - \rho\|_1 \leq N_A (\epsilon(\ell) + \gamma(\ell) + \delta(\ell))$$

If we can build the lattice $A^c$ with holes, then we can reconstruct the original lattice.
Step 2: Break up the connecting regions

\[ B_- \subset B \subset B_+ \]

- By the Markov condition

\[ \| R_{B+}^{\rho A_c} (\rho_{B_-}^{A_c}) - \rho_{B_-}^{A_c} \|_1 \leq N_B (\gamma(\ell) + \delta(\ell)) \]

- By Local indistinguishability

\[ \| \text{tr}_B [\rho^{(A_-B_-)^c}] - \rho_{B_-}^{A_c} \|_1 \leq N_B \epsilon(\ell) \]

Local cpt map \[ \mathcal{F}_B \equiv R_{B+}^{\rho A_c} \text{tr}_B \]

\[ \| \mathcal{F}_B \mathcal{F}_A (\rho^{(A_-B_-)^c}) - \rho \|_1 \leq (N_A + N_B)(\epsilon(\ell) + \gamma(\ell) + \delta(\ell)) \]

If we can build the lattice \((A_-B_-)^c\), then we can reconstruct the original lattice.
Step 3:

- Project onto $\rho^C$
- By locality
  $$F_C(\psi) = \rho^c \text{tr}_C[\psi]$$

Finally
  $$\|F_C F_B F_A(\psi) - \rho\|_1 \leq (N_C + N_A + N_B)(\epsilon(\ell) + \gamma(\ell) + \delta(\ell))$$

The entire lattice can be built from a local circuit of cpt maps.
GROUND STATES

Proof ingredients

- (uniform) Local indistinguishability
- (uniform) Markov condition
- Local definition of states

For injective PEPS, proof can be reproduced exactly.

We can show that the conditions of the theorem hold if the topological entanglement entropy is zero.
SPECTRAL GAP

We showed: \[ |\mathcal{F}_C \mathcal{F}_B \mathcal{F}_A (\psi) - \rho|_1 \leq L_D e^{-\ell/\xi} \]

Define \( \mathcal{F}_A = e^{t\mathcal{L}_A} \) \[ \mathcal{L}_A = \sum_j (\mathcal{F}_{A_j} - \text{id}) \]

If \( \mathcal{F}_A, \mathcal{F}_B, \mathcal{F}_C \) had the same fixed point, then is gaped, by the reverse detectability lemma.

The same strategy works for proving gaps of parent Hamiltonians of injective PEPS

New strategy for proving the gap of the 2D AKLT model!!!

All about boundary conditions
OUTLOOK

Spectral gap analysis, entanglement spectrum
  New classification for many-body systems

Approximate Quantum error correction
  Tradeoff bounds
  New codes?

Renormalization Group, critical models, AdS/CFT

S. Flammia, J. Haah, MJK, I. Kim, arXiv:1610.06169
THANK YOU!