The Niels Bohr International Academy

UNIVERSITY OF COPENHAGEN

# LOCAL RECOVERY MAPS AS DUCT TAPE FOR MANY BODY SYSTEMS

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November 14 2016, QuSoft Amsterdam

CARLSBERGFONDET

VILLUM FONDEN

# CONTENTS

# LOCAL RECOVERY MAPS

Strong subadditivity (SSA):

$$I_\rho(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC) \geq 0$$

Equality

$$\boxed{I_\rho(A:C|B) = 0 \quad \Leftrightarrow \quad R_{AB}(\rho_{BC}) = \rho}$$

Petz map

$$R_{AB}(\sigma) = \rho_{AB}^{1/2}\rho_B^{-1/2}\sigma\rho_B^{-1/2}\rho_{AB}^{1/2}$$

M. Ohya and D. Petz, (2004)

Markov State

$$\rho = \oplus_j \rho_{AB_j^L} \otimes \rho_{B_j^R C}$$

P. Hayden, et. al., CMP 246 (2004)

➡ there exists a disentangling unitary on B.

# Approximately LOCAL RECOVERY MAPS

## Strengthening SSA:

$$I_\rho(A:C|B) \geq -2\log_2 F(\rho, R_{AB}(\rho_{AB}))$$

O. Fawzi and R. Renner, CMP 340 (2015)

## Rotated Petz map

$$R_{AB}(\sigma) = \int dt \beta(t) \rho_{AB}^{\frac{1}{2}+it} \rho_B^{-\frac{1}{2}-it} \sigma \rho_B^{-\frac{1}{2}+it} \rho_{AB}^{\frac{1}{2}-it}$$

M. Junge, et. al. arXiv:1509.07127

ABC are arbitrary

- Is the map universal?
- Is the conditional mutual information necessary?
- Other properties of the map?

# APPLICATIONS

## Shannon Theory and Entanglement theory

# APPLICATIONS

Shannon Theory and Entanglement theory

Classical Simulations

Tensor networks, stoquastic models

Quantum Simulations (sampling)

Topological order

Quantum error correction

Renormalization Group, critical models, AdS/CFT

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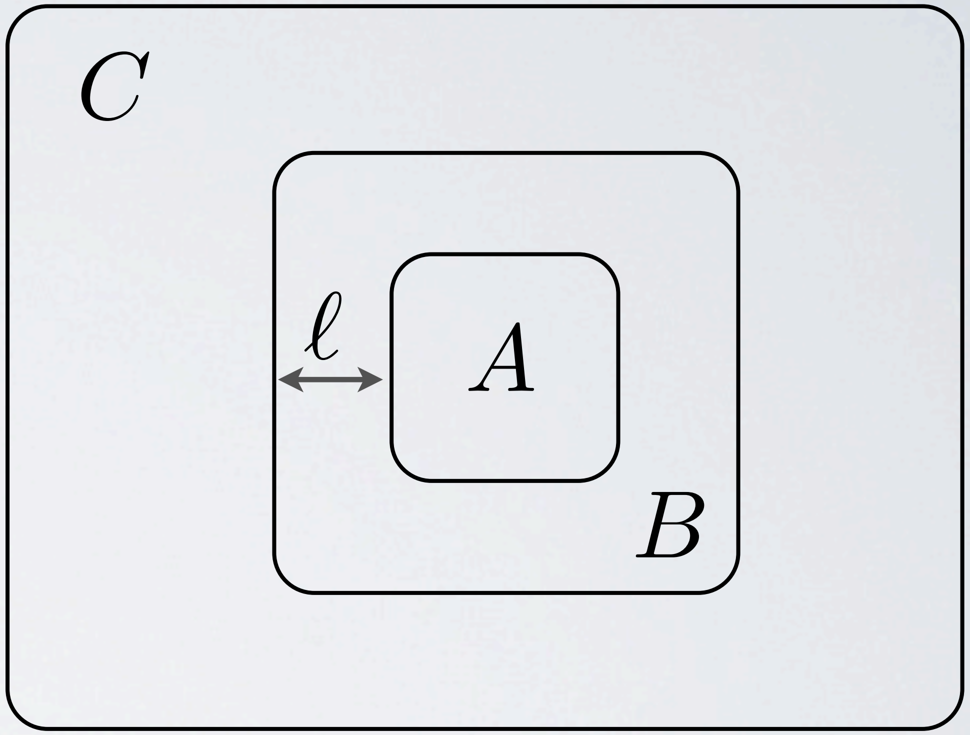
# MANY-BODY SETTING

## Exact recovery

For any A, and B shielding A:
$$I_\rho(A:C|B)=0$$

$$\mathcal{H}=\mathcal{H}_2^{\otimes N}$$



# HAMMERSLEY-CLIFFORD

## Exact recovery

For any A, and B shielding A:

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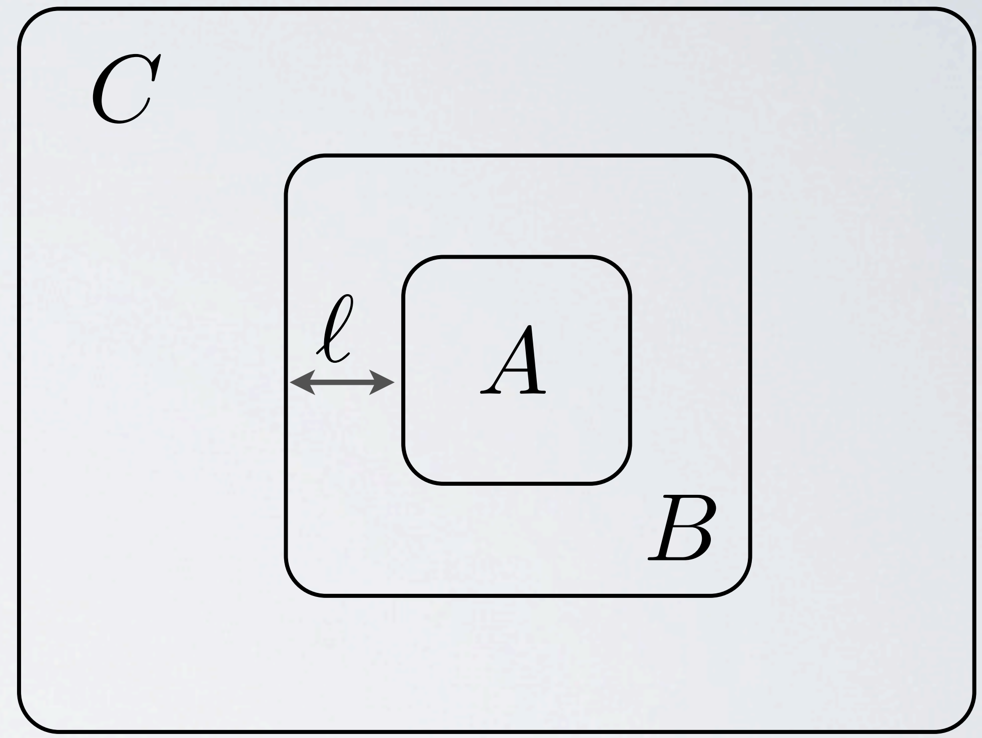
$\Leftrightarrow$ $\rho>0$ is the Gibbs state of a local commuting H

W. Brown, D. Poulin, arXiv:1206.0755

$\Leftrightarrow$ $\rho=|\psi\rangle\langle\psi|$ is the ground state of a local commuting H



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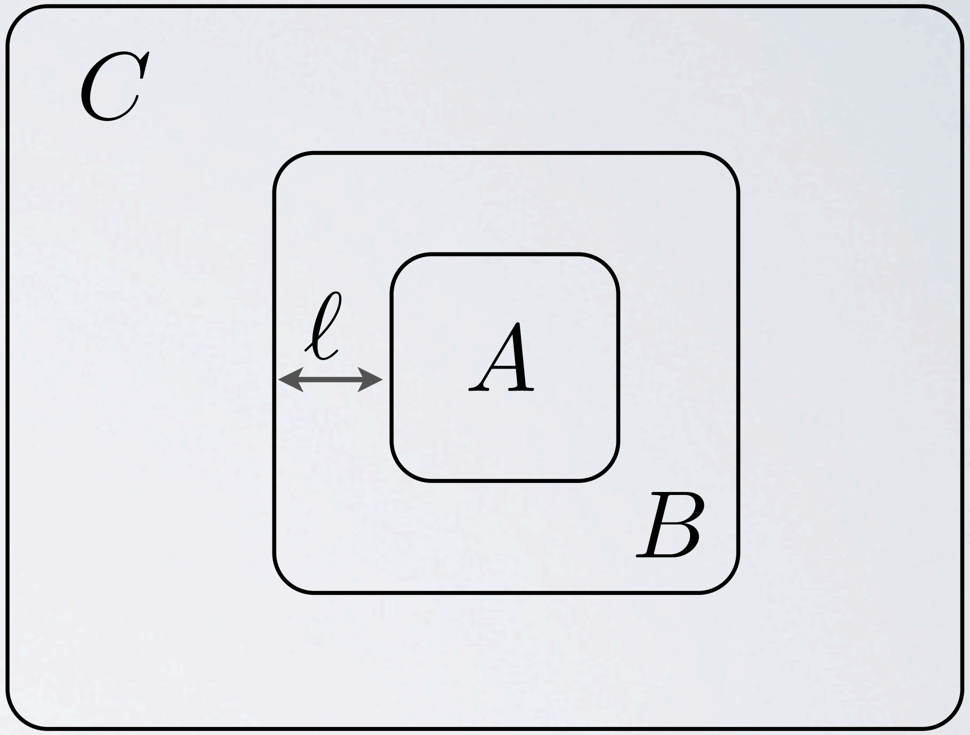


## Approximate recovery

For any A, and B shielding A: $I_\rho(A : C|B) \leq K e^{-c\ell}$

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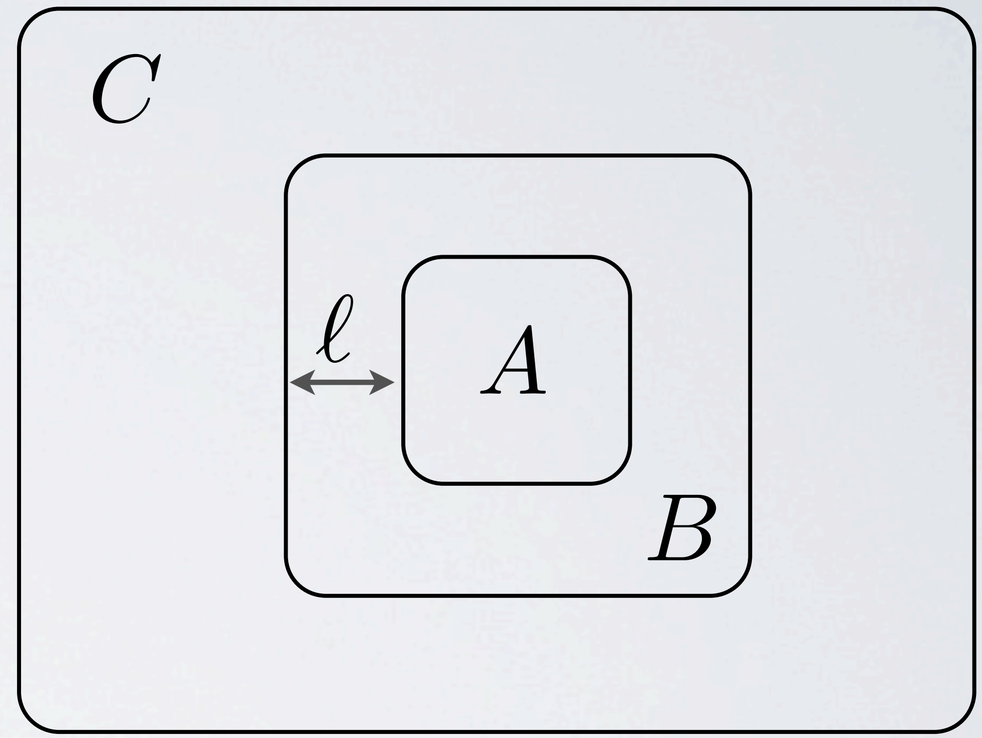



## Approximate recovery

For any A, and B shielding A: $I_\rho(A:C|B)\le Ke^{-c\ell}$

$\not\Leftrightarrow$ $\rho>0$ is the Gibbs state of a local non-commuting H

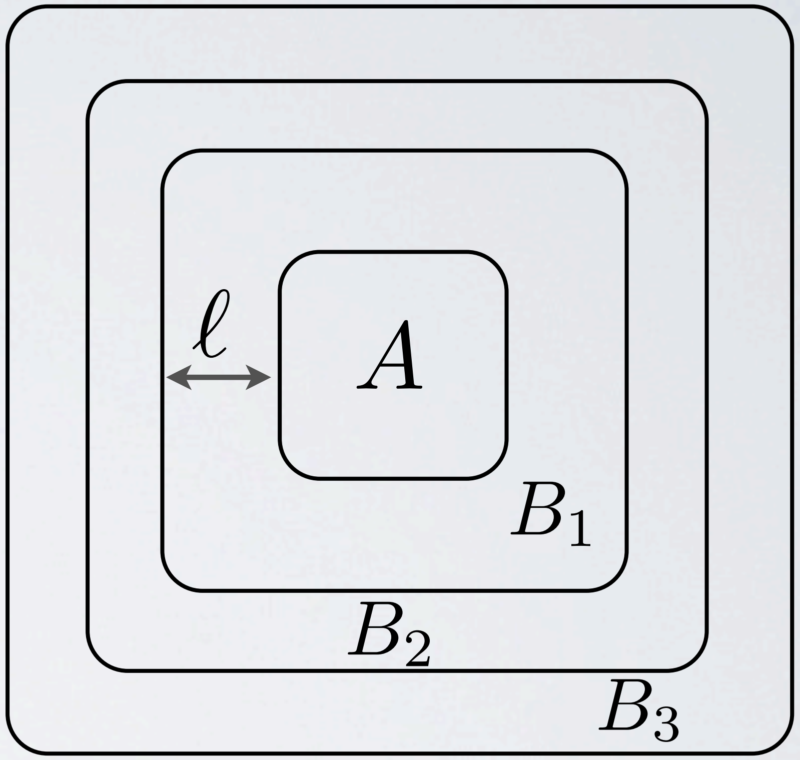
K. Kato, F Brandao, arXiv:1609.06636

$\not\Leftrightarrow$ $\rho=|\psi\rangle\langle\psi|$ is the ground state of a gaped local non-commuting H

# AREA LAW

## Further consequences

$$I(A:B_1\cdots B_{n+1}) - I(A:B_1\cdots B_n) = I(A:B_{n+1}|B_1\cdots B_n)$$




## Mutual info area law: $I(A:A^c) \leq c|\partial A|$

→ Decaying CMI provides a quantitative MI area law

# AREA LAW

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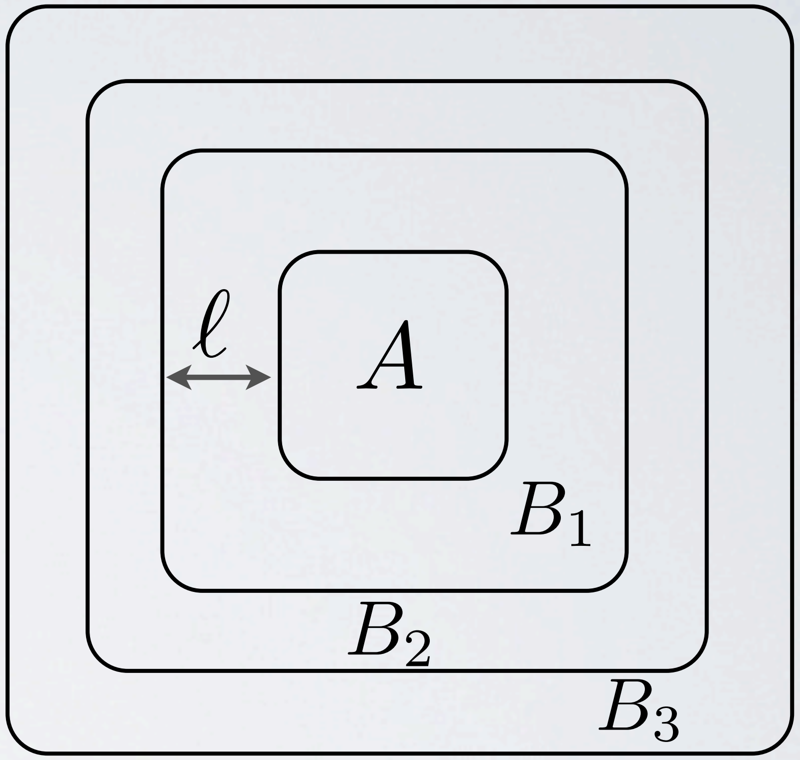


Decaying CMI provides a quantitative MI area law

**Can also show:** Small CMI implies efficient MPS/MPO representation!

**Take-home message:** CMI replaces Area Law, HC program replaces the area law conjecture

# AREA LAW

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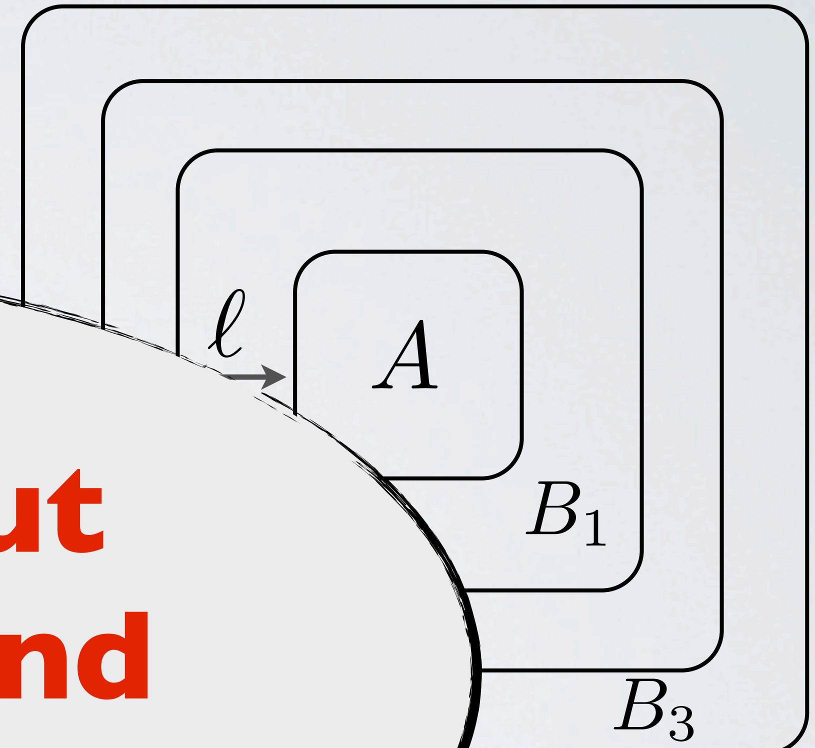


Decaying CMI provi

quantitati

**What about dynamics and state preparation?**

## Can also show

ation!

## Take-home message:

CMI replaces Area Law, HC program replaces the area law conjecture

# MONTE-CARLO SIMULATIONS

Want to evaluate: $\langle Q \rangle = \sum_x \pi(x) Q(x)$ $\pi \propto e^{-\beta H}$

classical Gibbs state

Idea: - obtain a sample configuration from the distribution $\pi$

- Set up a Markov chain with $\pi$ as an approximate fixed point

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Metropolis algorithm: (- start with random configuration)

- Flip a spin at random, calculate energy
- If energy decreased, accept the flip
- If energy increased, accept the flip with probability $p_{\text{flip}} = e^{-\beta \Delta E}$
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**Equilibrium?**

# ANALYTIC RESULTS

Note: - Glauber dynamics (Metropolis) is modeled by a semigroup $P_t = e^{tL}$

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Note: - Glauber dynamics (Metropolis) is modeled by a semigroup

$$P_t = e^{tL}$$

Fundamental result for Glauber dynamics:

$\pi$ has exponentially decaying correlations $\Leftrightarrow$ $P_t$ mixes in time $O(\log(N))$

$L$ is gapped

F. Martinelli, Lect. Prof. Theor. Stats , Springer
A. Guionnet, B. Zegarlinski, Sem. Prob., Springer

- independent of boundary conditions in 2D
- independent of specifics of the model
- no intermediate mixing

# QUANTUM GIBBS SAMPLERS

## Commuting Hamiltonian

Davies maps are another generalization of Glauber dynamics

MJK and K. Temme, arXiv:1505.07811

$$T_t = e^{t\mathcal{L}}$$

$$\mathcal{L} = \sum_{j\in\Lambda}(R_{j\partial} - id)$$

$R_{j\partial}$ is the Petz recovery map!

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MJK and F. Brandao, CMP 344 (2016)

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The exists a partial extension of the **statics = dynamics** theorem ✸

MJK and F. Brandao, CMP 344 (2016)

## Non-commuting Hamiltonian

$$\mathcal{L} = \sum_{j\in\Lambda}(R_{j\partial} - id)$$

$R_{j\partial}$ is the rotated Petz map!

⇒ no longer frustration-free

⇒ Theorem ✸ does not hold

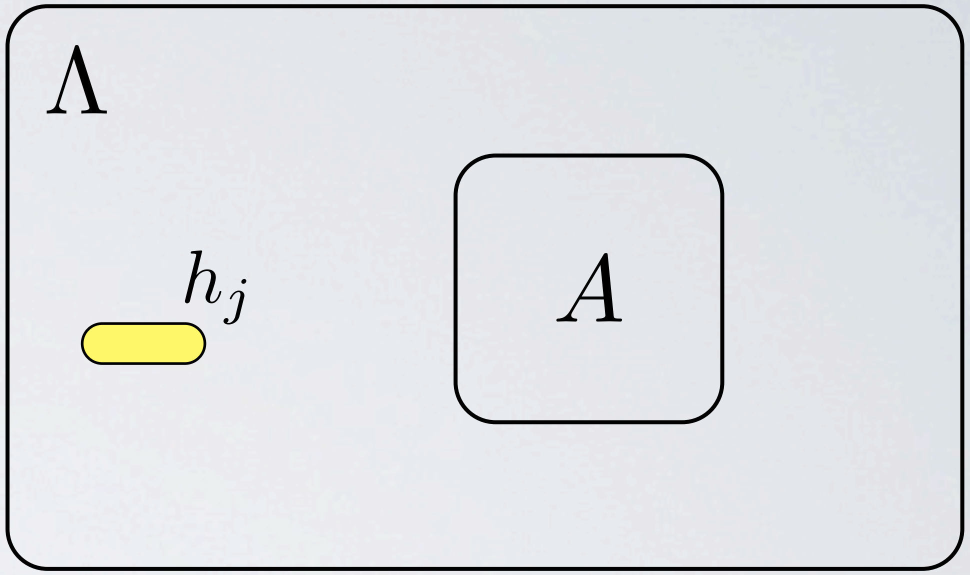
⇒ Davies maps are non-local

# STATE PREPARATION

Based on : MJK, F. Brandao, arXiv:1609.07877

# SETTING

Lattice: $A \subset \Lambda$




Hamiltonian: $H_A = \sum_{Z \subset A} h_Z$

$h_Z = 0 \text{ for } |Z| \geq K$

Gibbs states: $\rho^A = e^{-\beta H_A} / \mathrm{Tr}[e^{-\beta H_A}]$ is the Gibbs state restricted to A

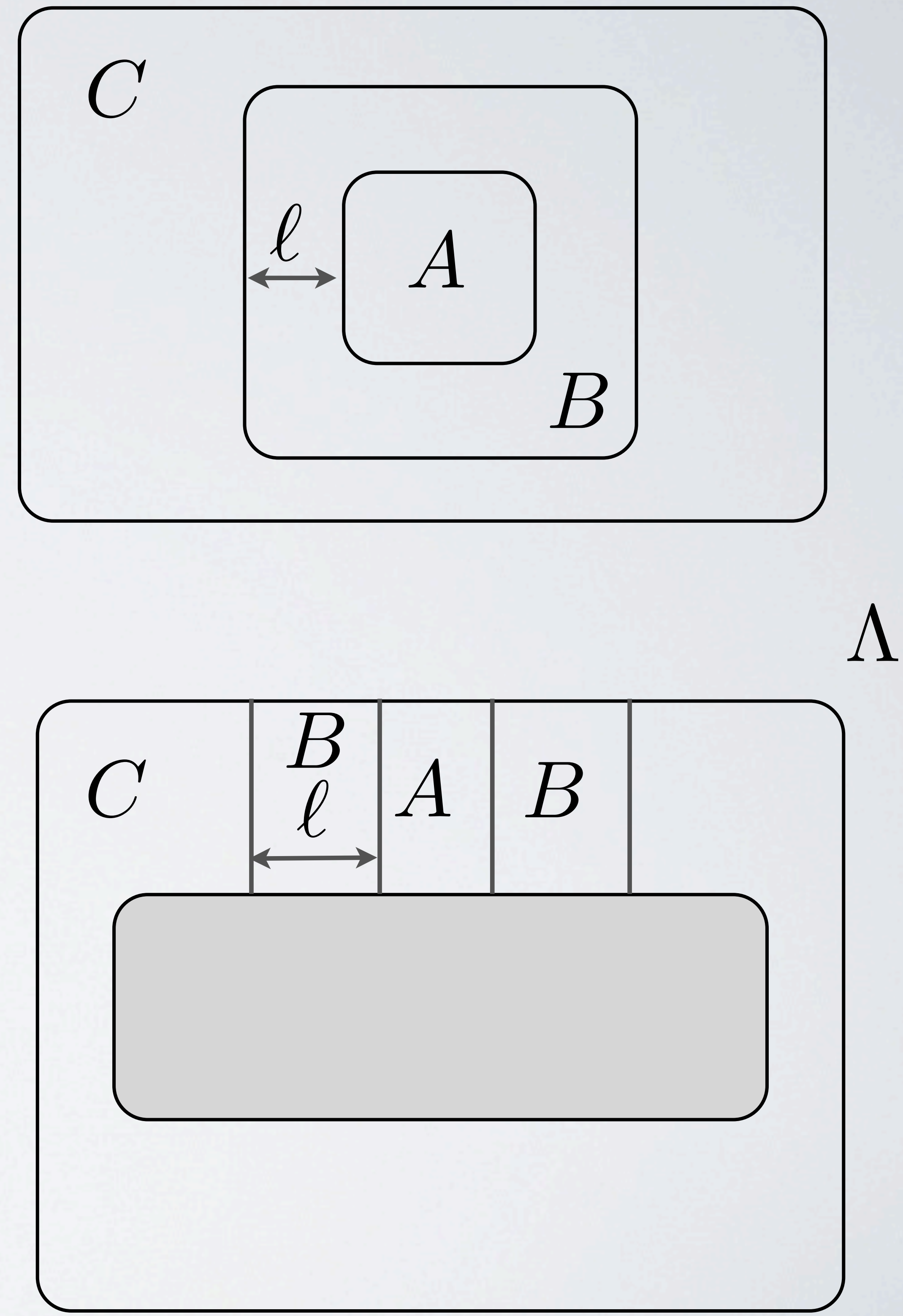
Note: Superscript for domain of definition of Gibbs state, while subscript for partial trace.

# THE MARKOV CONDITION

Uniform Markov:

Any subset $X = ABC \subset \Lambda$ with $B$ shielding $A$ from $C$ in $X$ , we have

$$I_{\rho^X}(A : C|B) \leq \delta(\ell)$$




Recall: $\rho^X = e^{-\beta H_X} / \mathrm{Tr}[e^{-\beta H_X}]$

Also must hold for non-contractible regions

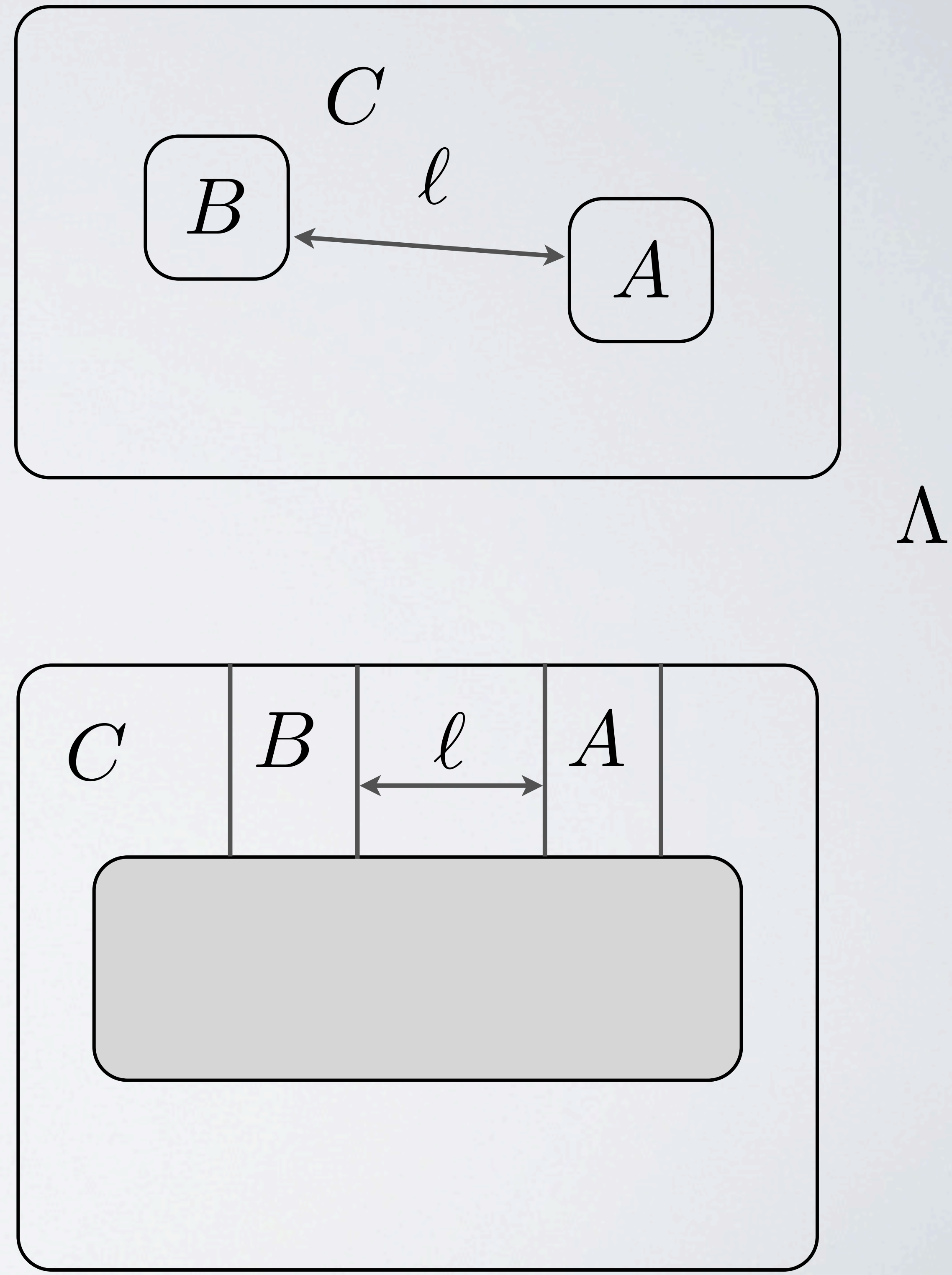
# CORRELATIONS

## Uniform Clustering:

Any subset $X = ABC \subset \Lambda$ with $\mathrm{supp}(f) \subset A$ and $\mathrm{supp}(g) \subset B$

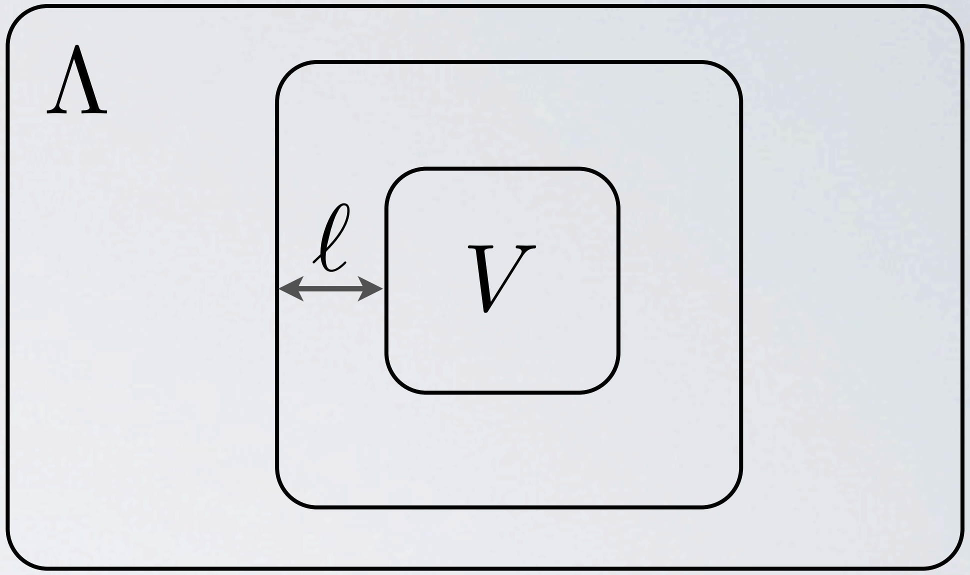
$$\mathrm{Cov}_{\rho^X}(f, g) \leq \epsilon(\ell)$$

$$\mathrm{Cov}_\rho(f, g) = |\mathrm{tr}[\rho f g] - \mathrm{tr}[\rho f]\mathrm{tr}[\rho g]|$$




Note: Uniform Clustering follows from uniform Gap

# LOCAL PERTURBATIONS




## Commuting Hamiltonian

$$e^{-\beta(H^A+H^B)} = e^{-\beta H^A} e^{-\beta H^B}$$

if $\quad [H^A, H^B] = 0$

## Non-commuting Hamiltonian

MB. Hastings, PRB 201102 (2007)

General $\quad e^{-\beta(H+V)} = O_V e^{-\beta H} O_V^\dagger$

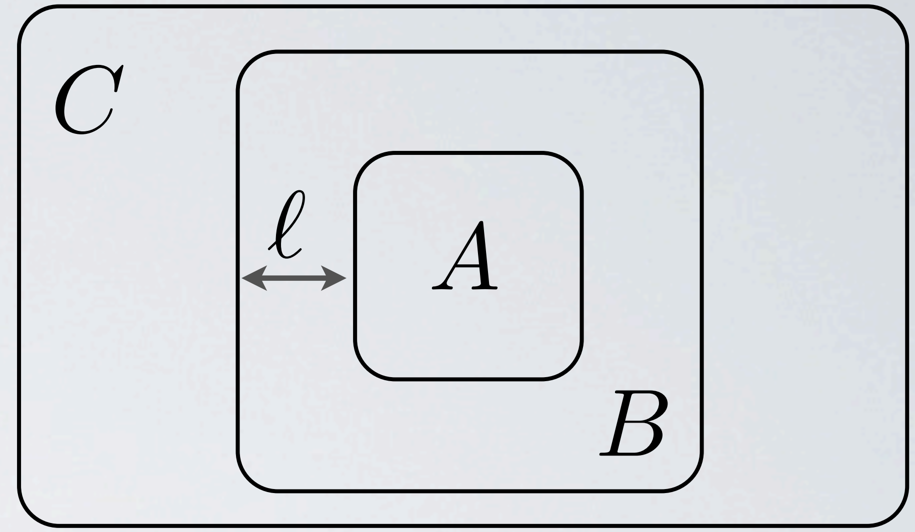
$$||O_V - O_V^\ell|| \leq c_1 e^{-c_2 \ell} \equiv \gamma(\ell) \qquad ||O_V|| \leq e^{\beta ||V||}$$

Only works if $V$ is local!

# APPROXIMATIONS

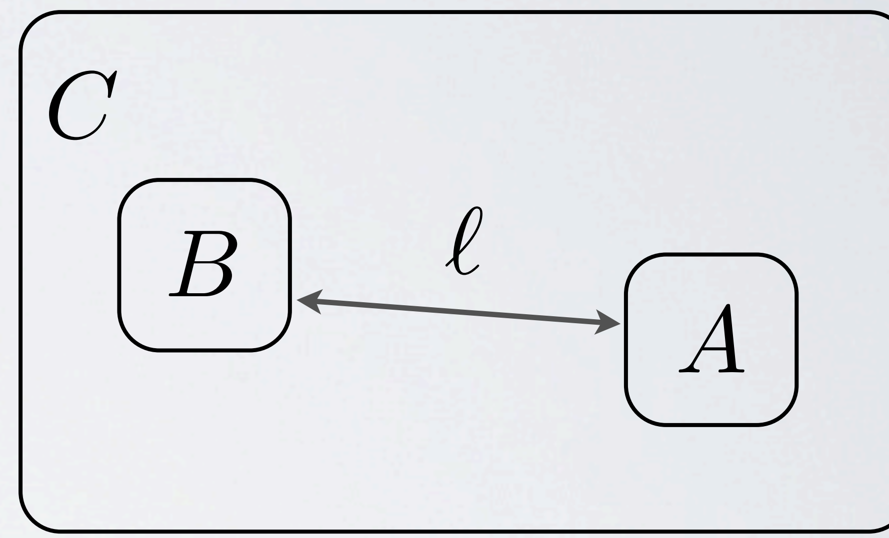
## Uniform Markov

$$I_{\rho^X}(A : C|B) \leq \delta(\ell)$$



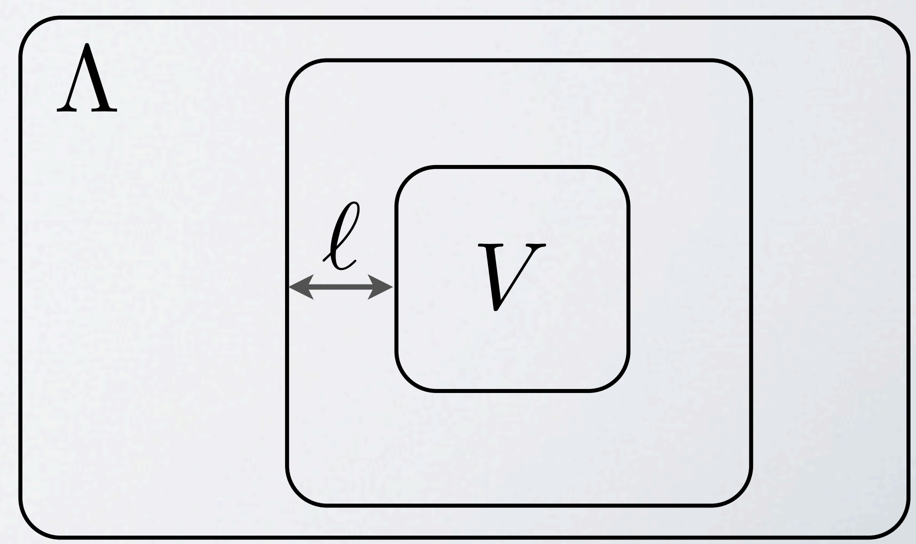

## Uniform clustering

$$\mathrm{Cov}_{\rho^X}(f, g) \leq \epsilon(\ell)$$




## Local perturbations

$$||e^{-\beta(H+V)} - O_V^{\ell} e^{-\beta H} O_V^{\ell}|| \leq c_1 e^{-c_2 \ell} \equiv \gamma(\ell)$$

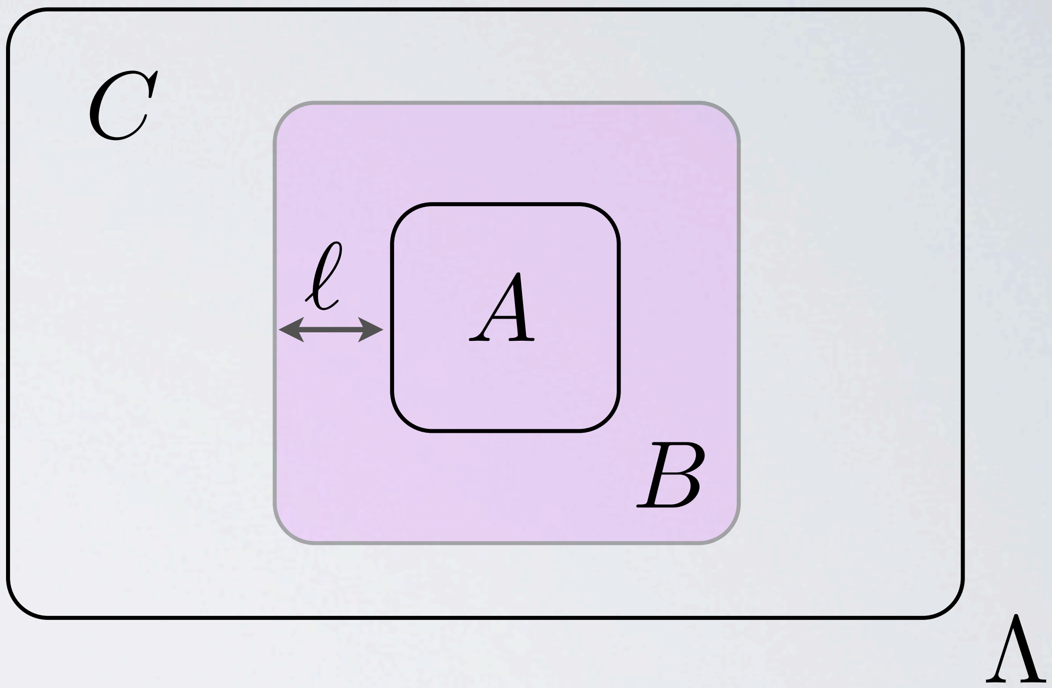


# LOCAL INDISTINGUISHABILITY

**Result 1:**

Any subset $X = ABC \subset \Lambda$ with $B$ shielding $A$ from $C$ in $X$, if $\rho$ is uniformly clustering,

$$||\mathrm{tr}_{BC}[\rho^{ABC}] - \mathrm{tr}_B[\rho^{AB}]||_1 \leq c|AB|(\epsilon(\ell) + \gamma(\ell))$$




Consequence: Efficient evaluation of local expectation values

$$\langle O_A \rangle = \mathrm{tr}[\rho^\Lambda O_A] \approx \mathrm{tr}[\rho^{AB} O_A]$$

# LOCAL INDISTINGUISHABILITY

**Result 1:**

Any subset $X = ABC \subset \Lambda$ with $B$ shielding $A$ from $C$ in $X$, if $\rho$ is uniformly clustering,

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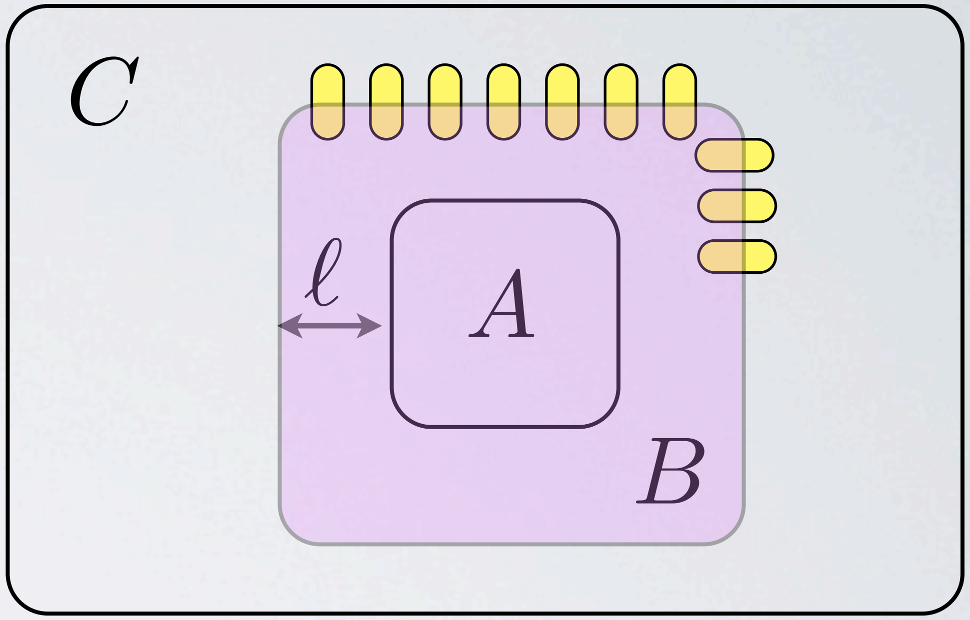
Proof idea:




Remove pieces of the boundary of $B$ one by one

telescopic sum $$||\mathrm{tr}_{BC}[\rho^X - \rho^{AB} \otimes \rho^C]||_1 \leq \sum_j ||\mathrm{tr}_{BC}[\rho^{X_{j+1}} - \rho^{X_j}]||_1$$

Bound each term $$||\mathrm{tr}_{BC}[\rho^{X_{j+1}} - \rho^{X_j}]||_1 \approx \sup_{g_A} |\mathrm{tr}[g_A(O_j^\ell \rho^{X_j} O_j^{\ell,\dagger} - \rho^{X_j}]|$$
$$= \mathrm{Cov}_{\rho^{X_j}}(g_A, O_j^{\ell,\dagger} O_j^\ell)$$

# STATE PREPARATION

**Main Result:**

If $\rho$ is uniformly clustering and uniformly Markov, then there exists a depth $D+1$ circuit of quantum channels $\mathbb{F} = \mathbb{F}_{D+1} \cdots \mathbb{F}_1$ of local range $O(\log(L))$, such that

$$\boxed{||\mathbb{F}(\psi) - \rho||_1 \leq cL^D(\epsilon(\ell) + \delta(\ell) + \gamma(\ell))}$$

MJK, F. Brandao, arXiv:1609.07877

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If $\rho$ is uniformly clustering and uniformly Markov, then there exists a depth $D+1$ circuit of quantum channels $\mathbb{F} = \mathbb{F}_{D+1} \cdots \mathbb{F}_1$ of local range $O(\log(L))$, such that

$$||\mathbb{F}(\psi) - \rho||_1 \leq cL^D(\epsilon(\ell) + \delta(\ell) + \gamma(\ell))$$

MJK, F. Brandao, arXiv:1609.07877

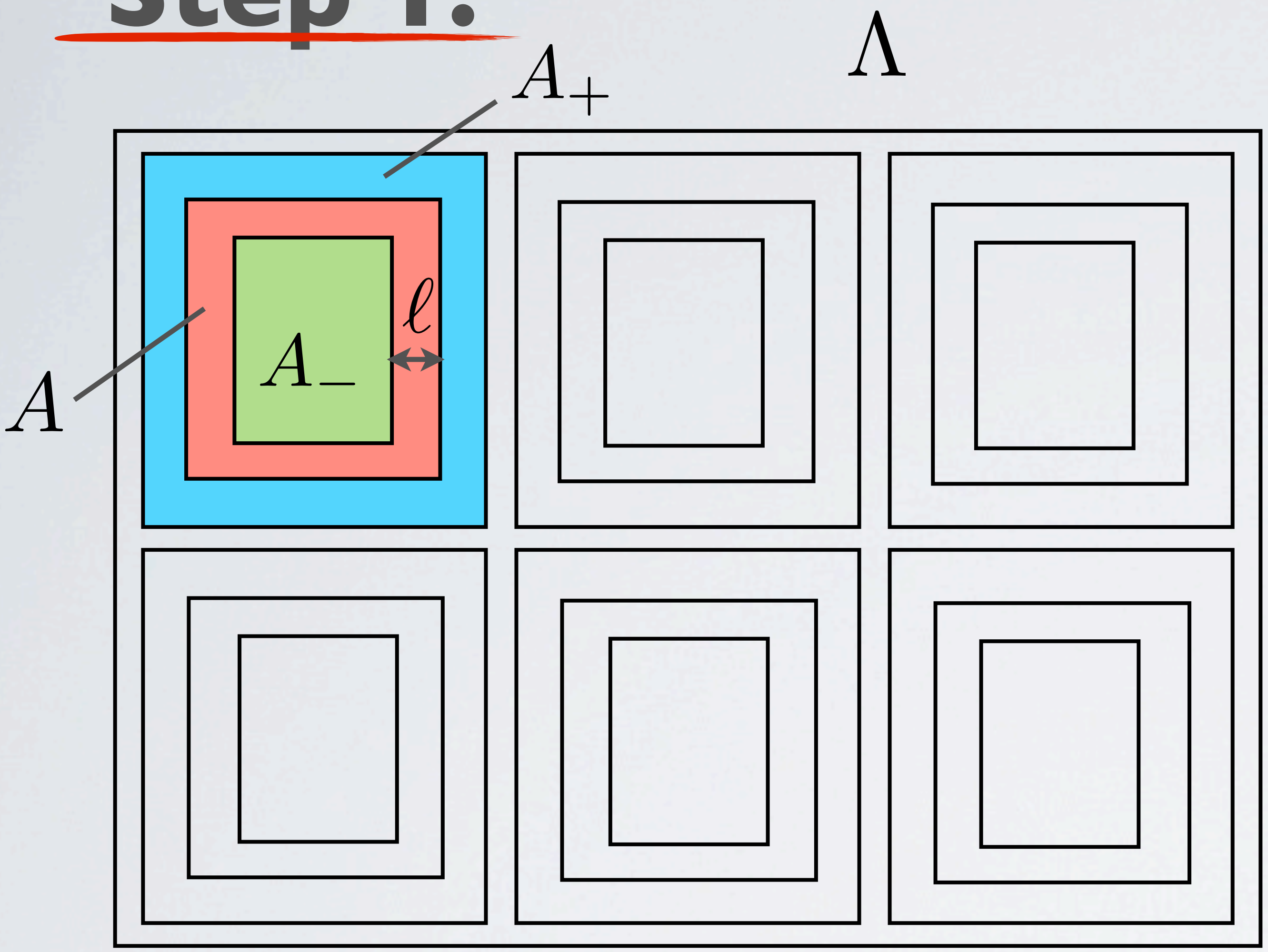
**Corollary:**

If $\rho$ is uniformly clustering and uniformly Markov, then there exists a depth $M = O(\log(L))$ circuit of strictly local quantum channels $\mathbb{F} = \mathbb{F}_M \cdots \mathbb{F}_1$, such that

$$||\mathbb{F}(\psi) - \rho||_1 \leq cL^D(\epsilon(\ell) + \delta(\ell) + \gamma(\ell))$$

# PROOF OUTLINE

**Step I:**




- Cover the lattice in concentric squares $A_- \subset A \subset A_+$
- By the Markov condition

$$||R^{\rho}_{A_+}(\rho_{A^c}) - \rho||_1 \leq N_A(\gamma(\ell) + \delta(\ell))$$

- By Local indistinguishability

$$||\mathrm{tr}_A[\rho^{A^c_-}_{A^c}] - \rho_{A^c}]||_1 \leq N_A \epsilon(\ell)$$

- Local cpt map $\mathbb{F}_A \equiv R^{\rho}_{A_+} \mathrm{tr}_A$

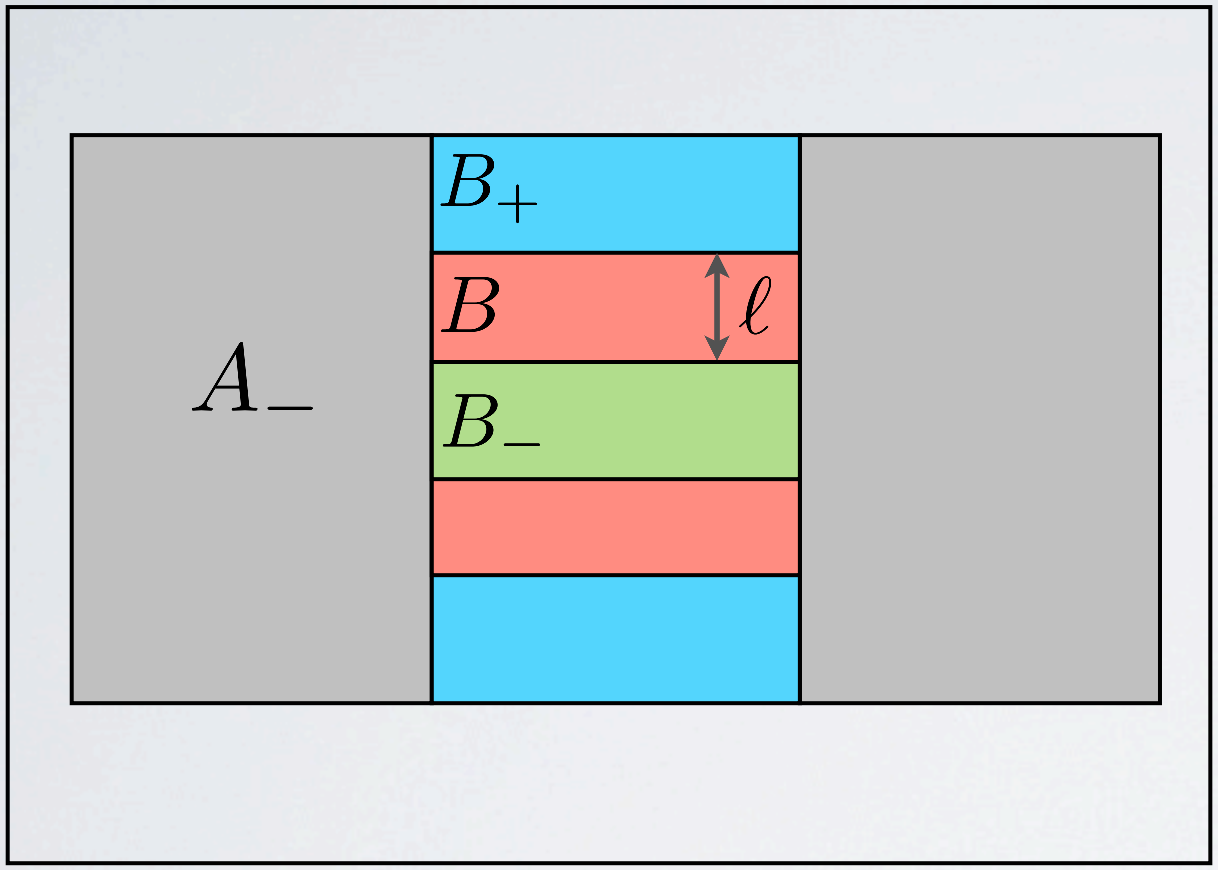
$$||\mathbb{F}_A(\rho^{A^c_-}) - \rho||_1 \leq N_A(\epsilon(\ell) + \gamma(\ell) + \delta(\ell))$$

➡ If we can build the lattice $A^c_-$ with holes, then we can reconstruct the original lattice.

# PROOF OUTLINE

**Step 2:**

$\Lambda$




- Break up the connecting regions

$$B_- \subset B \subset B_+$$

- By the Markov condition

$$||R_{B_+}^{\rho^{A_-^c}}(\rho_{B^c}^{A_-^c}) - \rho^{A_-^c}||_1 \leq N_B(\gamma(\ell) + \delta(\ell))$$

- By Local indistinguishability

$$||\mathrm{tr}_B[\rho^{(A_-B_-)^c}] - \rho_{B_-^c}^{A_-^c}]||_1 \leq N_B \epsilon(\ell)$$

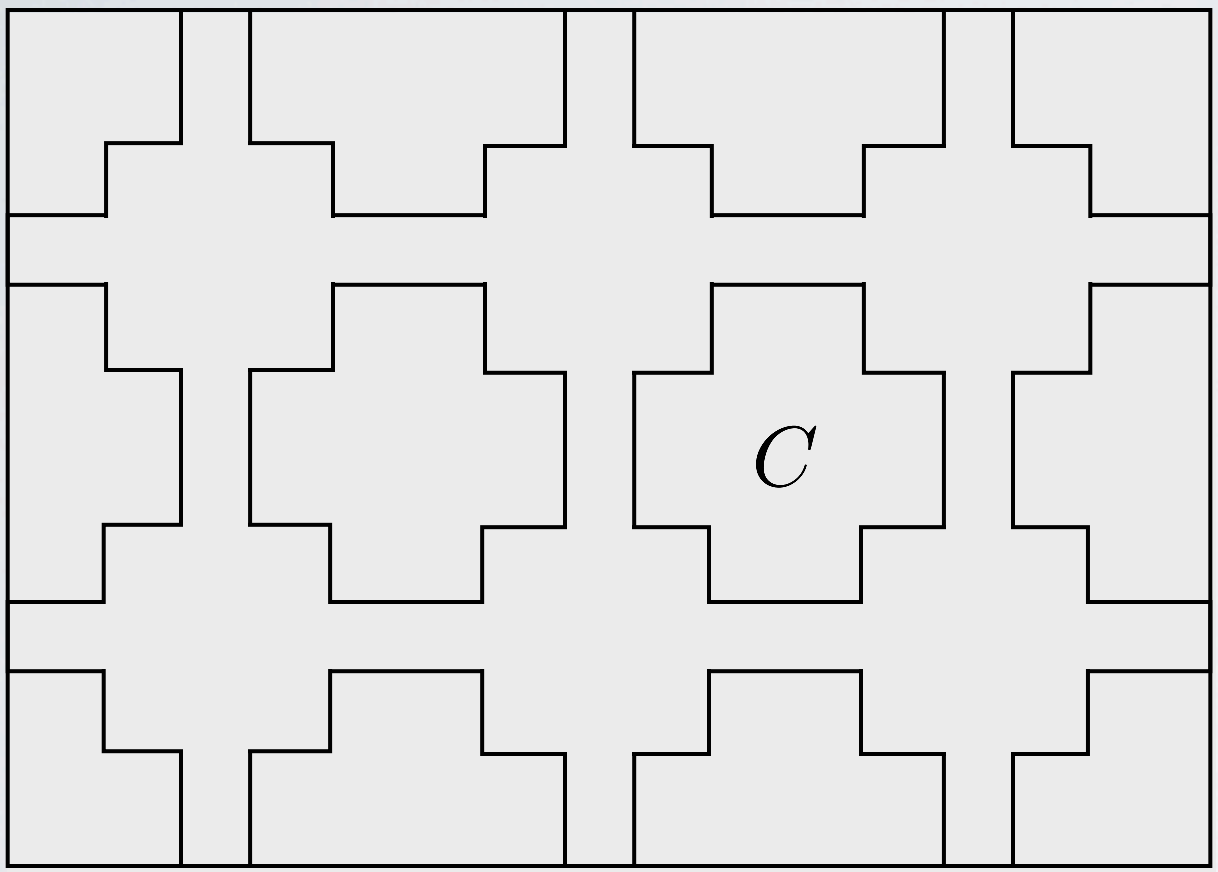
- Local cpt map $\mathbb{F}_B \equiv R_{B_+}^{\rho^{A_-^c}} \mathrm{tr}_B$

$$||\mathbb{F}_B \mathbb{F}_A(\rho^{(A_-B_-)^c}) - \rho||_1 \leq (N_A + N_B)(\epsilon(\ell) + \gamma(\ell) + \delta(\ell))$$

If we can build the lattice $(A_-B_-)^c$, then we can reconstruct the original lattice.

# PROOF OUTLINE

## Step 3:




- Project onto $\rho^C$
- By locality

$$\mathbb{F}_C(\psi) = \rho^c \mathrm{tr}_C[\psi]$$

- Finally $||\mathbb{F}_C\mathbb{F}_B\mathbb{F}_A(\psi) - \rho||_1 \leq (N_C + N_A + N_B)(\epsilon(\ell) + \gamma(\ell) + \delta(\ell))$

➡ The entire lattice can be built from a local circuit of cpt maps.

# GROUND STATES

Proof ingredients

- (uniform) Local indistinguishability
- (uniform) Markov condition
- Local definition of states

➡ For injective PEPS, proof can be reproduced exactly.

➡ We can show that the conditions of the theorem hold it the topological entanglement entropy is zero.

# SPECTRAL GAP

We showed: $||\mathbb{F}_C\mathbb{F}_B\mathbb{F}_A(\psi) - \rho||_1 \leq L^D e^{-\ell/\xi}$

Define $\mathbb{F}_A = e^{t\mathcal{L}_A}$ $\mathcal{L}_A = \sum_j (\mathbb{F}_{A_i} - \mathrm{id})$

If $\mathbb{F}_A, \mathbb{F}_B, \mathbb{F}_C$ had the same fixed point, then $\mathcal{L} = \mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C$ is gaped, by the reverse detectability lemma.

A. Anshu, et. al., Phys. Rev. B 93, 205142 (2016)

- The same strategy works for proving gaps of parent Hamiltonians of injective PEPS
- New strategy for proving the gap of the 2D AKLT model!!!

All about boundary conditions

# OUTLOOK

## Spectral gap analysis, entanglement spectrum

New classification for many-body systems

## Approximate Quantum error correction

Tradeoff bounds

S. Flammia, J. Haah, MJK, I. Kim, arXiv:1610.06169

New codes?

## Renormalization Group, critical models, AdS/CFT

THANK YOU!