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Markov Processes and Related Fields

Persistence of non-Markovian Processes Related to Fractional Brownian Motion

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Abstract. The persistence probability $P(T)$ of a stochastic process $X(t)$ is defined as the probability that X does not change sign during a time interval of length T. Here it is pointed out that the asymptotic decay rate of $P(T)$ can be determined exactly for a one-parameter family of stationary Gaussian processes, which are related to fractional Brownian motion (fBm). The argument consists of two steps. First, the asymptotic behavior $P(T) \sim T^{-(1-H)}$ is derived for fBm with Hurst exponent H, a result which is argued to hold for any translationally invariant, self-affine process with stationary increments whose variance increases as $|t-t'|^{2H}$. Second, fBm is mapped onto a stationary process by going to logarithmic time. Through a comparison theorem of Slepian, our result can be used to obtain bounds on the persistence probability of more general processes.

KEYWORDS: Gaussian processes, first passage times, level crossings, fractional Brownian motion

AMS SUBJECT CLASSIFICATION: Primary 60G17, 60G18, 82C41

1. Introduction

Consider a stationary Gaussian stochastic process $X(t)$ with zero mean, unit variance and covariance function $\langle X(t)X(0)\rangle = f(t)$, $f(0) = 1$. We define the *persistence probability*

$$P(T \mid X(t)) \equiv \mathrm{Prob}[X(t) \neq 0, 0 \leq t \leq T] \tag{1.1}$$

as the probability that the process does not cross zero in a time interval of length T. In the cases of interest here P decays exponentially for large T,

$$P(T \mid X(t)) \sim e^{-\theta T}, \quad T \to \infty. \tag{1.2}$$

The key question, then, is how to determine the decay rate θ from the knowledge of $f(t)$. In full generality, this is a difficult and unsolved problem [4,5,17].

If X is Markovian, then $f(t) = \exp\{-\lambda|t|\}$ and (see [17])

$$P(T \mid X(t)) = \frac{2}{\pi} \arcsin e^{-\lambda T}, \tag{1.3}$$

hence $\theta = \lambda$. For non-Markovian processes, the persistence probability is explicitly known only in a few special cases [4,17]. Here I will show that for a one-parameter family of processes $X_H(t)$ governed by the covariance functions

$$f_H(t) = \cosh(Ht) - \frac{1}{2}|2\sinh(t/2)|^{2H}, \quad H \in (0,1), \tag{1.4}$$

the persistence decay rate is given by the simple expression

$$\theta_H = 1 - H. \tag{1.5}$$

In two special cases this relation is obvious. For $H = 1/2$, (1.4) reduces to $f_{1/2}(t) = \exp\{-|t|/2\}$, hence $X_{1/2}$ is Markovian and $\theta_{1/2} = \lambda = 1/2$. On the other hand, for $H = 1$ we have $f_1(t) \equiv 1$, so $X_1(t)$ is the constant process with $\theta_1 = 0$. The general relation (1.5) will be derived in Section 2 by mapping the $X_H(T)$ to fractional Brownian motion [14] and exploiting the scale invariance of the latter. In Section 3 it is shown how bounds on the persistence probability for a larger class of processes can be obtained using (1.5) and a theorem due to Slepian [17].

The relation (1.5) has been presented in a previous publication concerned with the persistence of fluctuating interfaces [9]. The purpose of this note is to give a more careful and critical derivation, and to make the result available to a probabilistic audience. The paper contains no mathematical proofs, and thus its statements should be regarded as (hopefully plausible) conjectures. Rigorous versions of the arguments would be most welcome.

2. Fractional Brownian motion

2.1. Definition and basic properties

Fractional Brownian motion $B_H(s)$ is a Gaussian process with stationary increments of mean zero and variance

$$\langle |B_H(s) - B_H(s')|^2 \rangle = |s - s'|^{2H} \tag{2.1}$$

where $H \in (0,1)$ is the *Hurst exponent* [13,14]. Since the process is invariant under shifts $B_H(s) \to B_H(s) + const$, it is convenient to fix its value at $s = 0$,

$$B_H(0) = 0. \tag{2.2}$$

For $H = 1/2$, $B_H(s)$ is Markovian and reduces to the standard Wiener process. Fractional Brownian motions with $H \neq 1/2$ arise naturally in the Markovian description of spatially extended systems, where the dynamics at a fixed point in space becomes non-Markovian due to the coupling to the neighborhood. For example, $B_{1/4}(s)$ describes the fluctuations of certain one-dimensional interfaces [9] or harnesses [18] driven by Gaussian noise.

In contrast to the processes considered in Section 1, $B_H(s)$ is not stationary. The definition of the persistence probability therefore must contain two time arguments,

$$P_H(S_0, S_1) \equiv \text{Prob}[B_H(s) \neq 0, 0 < S_0 < s < S_1]. \tag{2.3}$$

For the Wiener process $B_{1/2}(s)$ a simple calculation yields the explicit expression

$$P_{1/2}(S_0, S_1) = \frac{2}{\pi} \arcsin \left[\sqrt{S_0/S_1}\right]. \tag{2.4}$$

Fractional Brownian motion is *self-affine* [13], i.e. the process is statistically invariant under the rescaling

$$B_H(s) \to b^{-H} B_H(bs) \tag{2.5}$$

where $b > 0$ is an arbitrary scale factor. Consequently the persistence probability can depend only on the ratio of the two time arguments,

$$P_H(S_0, S_1) = F_H(S_1/S_0), \tag{2.6}$$

as is illustrated by the result (2.4) for the Markovian case. By definition $F_H(1) = 1$. Below I will argue that for general H

$$P_H(S_0, S_1) \sim (S_0/S_1)^{1-H}, \quad S_1 \gg S_0. \tag{2.7}$$

While this result does not seem to be rigorously established, heuristic arguments have been presented in several papers [7,8]. The derivation given in the following section appears to be simpler and more transparent than earlier versions.

2.2. Geometry of zero crossings

The power law dependence of the variance (2.1) on the time increment implies that the set of zero crossings of $B_H(s)$ has Hausdorff dimension [1,3,13,15]

$$D_H = 1 - H. \tag{2.8}$$

Usually this is taken to imply that the number $N(\varepsilon, L)$ of intervals of size ε required to cover the zero crossings in $[0, L]$ behaves as

$$N(\varepsilon, L) \sim (L/\varepsilon)^{1-H} \tag{2.9}$$

for $\varepsilon \to 0$. When viewed at infinite resolution, the number of zero crossings in any finite interval is infinite. However, since we are interested in the persistence probability at large times, that is, the occurrence of large gaps between subsequent crossings, a fixed finite resolution scale ε suffices [2]. Due to the scale invariance property (2.5) the relation (2.9) holds equally well for fixed ε and L, and determines then the expected (finite!) number of coarse-grained zero crossings which can be distinguished at this resolution.

Let us therefore chose ε and L with $L/\varepsilon \gg 1$, and ask how the number $N_>(\varepsilon, L)$ of gaps of sizes exceeding ε depends on ε. The answer is given by a simple renormalization argument. Initially we have a total of $M = L/\varepsilon$ covering intervals, a number $N = (L/\varepsilon)^{D_H}$ of which contain zero crossings, while the remaining $M - N$ intervals are empty. Now we double the resolution scale, $\varepsilon \to 2\varepsilon$, by merging subsequent pairs of covering intervals. When both intervals of length ε are empty (filled), the resulting interval of length 2ε will be empty (filled). If these were the only possibilities, the number of filled intervals would be halved, whereas in fact $N(2\varepsilon, L) = 2^{-D_H} N(\varepsilon, L) > (1/2)N(\varepsilon, L)$. The excess of filled intervals at scale 2ε comes from pairs of intervals at scale ε in which a filled interval sits next to an empty one. These pairs are located at the edges of sequences of empty intervals, that is, at the edges of gaps larger than ε. Since each gap has two edges, the number of such large gaps can be estimated as

$$N_>(\varepsilon, L) \approx (2^{-(D_H+1)} - 2^{-2})N(\varepsilon, L) \sim (L/\varepsilon)^{1-H}. \tag{2.10}$$

The probability of finding a gap of size larger than ε is proportional to $\varepsilon^{-(1-H)}$, in agreement with (2.7). Note that the argument suggests a general relation between the Hausdorff dimension and the distribution of gap sizes for self-similar sets on the line, which can be easily verified for the standard Cantor set [13].

The derivation relies on two properties of $B_H(s)$: its scale invariance (2.5), and the invariance of the increment statistics (2.1) under translations in time (the increments are *homogeneous* [1]). Together these two features imply the self-similarity and translational invariance of the set of zero crossings. In contrast, the Gaussian character of the process is not important. The power law (2.7) should therefore be valid also for non-Gaussian, self-affine, translationally invariant processes with Hurst exponent H. There is some numerical evidence in support of this conjecture [7, 16]. On the other hand, (2.7) generally fails for Gaussian, self-similar processes which are not translationally invariant in time, such as "deformed" fractional Brownian motions defined by

$$\langle |\tilde{B}_H(s) - \tilde{B}_H(s')|^2 \rangle = a_H(s/s')|s - s'|^{2H} \tag{2.11}$$

with some positive, bounded function a_H. The factor $a_H(s/s')$ does not affect the scaling symmetry (2.5) but it clearly destroys the translational invariance of the set of level crossings. Processes of this kind arise in the dynamics of fluctuating interfaces starting from a flat initial condition [9], see Section 3. Another example is the Riemann–Liouville version of fractional Brownian motion [10, 14].

2.3. The logarithmic time transformation

It remains to establish the relation between fractional Brownian motion and the stationary process $X_H(t)$ introduced in Section 1. To this end we first define, for $s > 0$, the process

$$Y_H(s) = s^{-H} B_H(s), \tag{2.12}$$

which has zero mean, unit variance, and the covariance function

$$\langle Y_H(s) Y_H(s') \rangle = \frac{1}{2}\left[\left(\frac{s}{s'}\right)^H + \left(\frac{s'}{s}\right)^H - \left|\sqrt{\frac{s}{s'}} - \sqrt{\frac{s'}{s}}\right|^{2H}\right]. \tag{2.13}$$

Since this depends only on the ratio s/s', the process $Y_H(s)$ becomes stationary by passing to logarithmic time $t = \ln s$ [6, 11, 12]. Defining

$$X_H(t) = Y_H(e^t) \tag{2.14}$$

the covariance (2.13) therefore reduces to (1.4), and the persistence probability of $X_H(t)$ becomes, using (2.6) and (2.7),

$$P(T|X_H(t)) = F_H(e^T) \sim e^{-(1-H)T}, \quad T \to \infty, \tag{2.15}$$

establishing our main result (1.5). In the Markovian case, $H = 1/2$, inserting the expression (2.4) in (2.15) yields (1.3).

3. Applications

The usefulness of a relation like (1.5) lies in a comparison theorem due to Slepian (Theorem 1 of [17]), which states that for two processes X, Y with covariance functions $f(t)$, $g(t)$, where $f(t) \geq g(t)$ in some interval $0 \leq t \leq T_0$, the corresponding persistence probabilities satisfy $P(T \mid X(t)) \geq P(T \mid Y(t))$ for $0 \leq T \leq T_0$. If $T_0 = \infty$, this yields an inequality between the asymptotic persistence decay rates of the two processes.

Slepian's theorem is particularly powerful if the two covariance functions have the same type of singularity near $t = 0$, because then both upper and lower bounds on the persistence of one process can be obtained in terms of the persistence of the other [17]. Following Slepian we define the *class* α of a covariance function $f(t)$ to be the leading power in an expansion around $t = 0$,

$$f(t) = 1 - C|t|^\alpha + o(|t|^\alpha). \tag{3.1}$$

From (1.4) we obtain $\alpha = 2H$ and $C = 1/2$ for the covariance function $f_H(t)$. For the application of the comparison theorem it is also useful to know the behavior of $f(t)$ for large t. We will assume an exponential decay with rate λ,

$$f(t) \sim e^{-\lambda t}, \quad t \to \infty. \tag{3.2}$$

The decay rate of $f_H(t)$ is given by

$$\lambda_H = \min[H, 1-H]. \tag{3.3}$$

Now let $f(t)$ be a monotonically decreasing covariance function of class $\alpha \in (0,2)$, with known values of the amplitude of the leading term in the short time expansion (3.1), C, and the asymptotic decay rate λ. It is always possible to find scale factors $b_{\min}$ and $b_{\max}$ such that

$$f_{\alpha/2}(b_{\max}t) \leq f(t) \leq f_{\alpha/2}(b_{\min}t) \tag{3.4}$$

for all t. Using (1.5), the unknown decay rate θ of the Gaussian process governed by $f(t)$ is then bounded by

$$b_{\min}(1-\alpha/2) \leq \theta \leq b_{\max}(1-\alpha/2). \tag{3.5}$$

To establish that (3.4) holds for all times the whole function $f(t)$ has to be examined. However simple bounds on $b_{\min}$ and $b_{\max}$ can be obtained using the short and long time asymptotics of f. Comparing (3.1) and (3.2) to the corresponding behavior of $f_H(t)$ one finds

$$\begin{aligned} b_{\max} &\geq \max[\lambda/\lambda_H, (2C)^{1/2H}], \\ b_{\min} &\leq \min[\lambda/\lambda_H, (2C)^{1/2H}], \end{aligned} \tag{3.6}$$

and with some luck these inequalities may be satisfied as equalities.

In [9] this method was applied to the one-parameter family of processes $\tilde{X}_H(t)$ governed by the covariance function

$$\tilde{f}_H(t) = [\cosh(t/2)]^{2H} - |\sinh(t/2)|^{2H}, \quad H \in (0,1), \tag{3.7}$$

which is the logarithmic time version of a deformed fractional Brownian motion as given by (2.11), with the amplitude function

$$a_H(x) = 1 + \frac{2^{2H-1}(1+x^{2H}) - (1+x)^{2H}}{|1-x|^{2H}}, \quad a_H(0) = 2^{2H-1}, \ a_H(1) = 1. \tag{3.8}$$

For $0 < H < 0.1366\ldots$ the bound

$$\tilde{\theta}_H \geq \frac{(1-H)^2}{H} \tag{3.9}$$

was established for the decay rate of $P(T \mid \tilde{X}_H(t))$, which illustrates the dramatic reduction of the persistence for small H due to the innocuous factor a_H in (2.11): while the decay rate for X_H tends to unity for $H \to 0$, the bound (3.9) forces $\tilde{\theta}_H$ to diverge in this limit. This can also be read off directly from the expression (3.7) for the covariance function, which vanishes for any nonzero t when $H \to 0$.

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Decay to Equilibrium in L^∞ of Finite Interacting Particle Systems in Infinite Volume

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Abstract. We consider n particles evolving as asymmetric random walks in the lattice $\mathbf{Z}^d$ with an exclusion rule that allows at most one particle per site. For two subsets A, B of $\mathbf{Z}^d$ with n elements, denote by $p_t(A, B)$ the probability for the system being at time t at B if it started from A. We prove that there exists a universal constant $C = C(d, n)$ such that $p_t(A, B) \le Ct^{-nd/2}$. In the case of mean zero gradient processes we obtain Gaussian estimates with logarithmic corrections of the off-diagonal terms.

Keywords: interacting particle systems, decay to equilibrium

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1. Introduction

One of the first questions in the study of Markov processes is the investigation of the rates of convergence to equilibrium. Several tools have been developed in this respect in these last years and among them estimates for the spectral gap of the generator of reversible processes and logarithmic Sobolev inequalities.

In the context of conservative interacting particle processes, a spectral gap for the generator of the symmetric simple exclusion process restricted to a finite cube has been obtained by Quastel [12], see also [5]. Lu and Yau [11] introduced a general method to prove a spectral gap for conservative and non conservative dynamics and applied the method to Kawasaki dynamics with mixing conditions. Landim, Sethuraman and Varadhan [9] applied Lu and Yau's method to zero range dynamics, where the spins are unbounded. On the other hand, recently Yau [13, 14] proved the logarithmic Sobolev inequality for generalized exclusion processes and Kawasaki dynamics with strong mixing conditions.