UNIVERSITÄT ZU KÖLN

Prof. Dr. Joachim Krug
Dr. Ivan Szendro
Institut für Theoretische Physik
Sommersemester 2012

10. Exercise sheet to the lecture “Statistical Physics Far from Equilibrium”

Exercise 23: Universal form of the height-difference correlation function

Using the approach explained in the lectures, determine the scale factor θ that makes the scaling function $\mathcal{G}$ in the relation

$$G(r,s) = \langle (h(x,t) - h(x+r,t+s))^2 \rangle = A|r|\mathcal{G}(|r|/|\theta s|^{3/2}) \tag{1}$$

manifestly universal (model-independent) for models in the KPZ universality class. Here $A = \frac{D}{2\nu}$ is the model-dependent parameter governing the stationary height distribution, and θ should be expressed in terms of A and the coefficient λ of the KPZ nonlinearity.

Exercise 24: Poly-nuclear growth I

In the one-dimensional PNG-model 'islands' of unit height and zero width nucleate randomly at rate Γ per unit time and length. Once created, the two edges of the island spread laterally at speed c. The state of the system can therefore be described by the number and positions of left- and right-moving steps. When two islands collide they merge, and correspondingly a pair of steps is annihilated.

a.) We assume (and will prove below in Exercise 25) that left- and right moving steps form an 'ideal gas' of uncorrelated particles moving at speed $\pm c$. Denoting the densitites of left- and right-moving steps by ρ_L and ρ_R, the balance between pairwise creation and annihilation in the steady state implies that

$$2c\rho_L\rho_R = \Gamma. \tag{2}$$

Moreover, the average slope u of the surface and the growth velocity V are related to ρ_L and ρ_R by

$$u = \rho_L - \rho_R, \quad V = c(\rho_L + \rho_R). \tag{3}$$

Explain why the relations (2,3) are true, and use them to compute the inclination-dependent growth velocity $V(u)$ for the one-dimensional PNG-model [compare also to Exercise 16 c.)].

b.) The PNG-model is easily generalized to $d+1$ dimensions[1]. Islands nucleate at rate Γ per unit time and d-dimensional area, and spread isotropically at speed c as d-dimensional spheres. Use dimensional analysis to show that

$$V(0) \sim (\Gamma c^d)^{\frac{1}{d+1}}. \tag{4}$$

[1] W. van Saarloos and G.H. Gilmer, Phys. Rev. B **33** (1986) 4927.

Exercise 25: Poly-nuclear growth II

We now consider the one-dimensional PNG model at zero slope ($u = 0$) but on a *finite* substrate of length L with periodic boundary conditions[2]. At $u = 0$, the numbers of left- and right-moving steps are equal; since steps are created and annihilated in pairs, this condition is preserved by the dynamics. The microscopic configuration $\mathcal{C}$ of the system is then determined by the positions $\mathcal{C} = \{x_1, ..., x_N, y_1, ..., y_N\}$ of N left-moving (x_i) and right-moving (y_i) steps; note that N is a fluctuating quantity. The probability of a given configuration is denoted by $P_N(\mathcal{C}, t)$. It satisfies the master equation

$$\frac{\partial}{\partial t} P_N(\mathcal{C}, t) = c \sum_{i=1}^{N} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial y_i} \right) P_N(\mathcal{C}, t) +$$

$$+2c \int_0^L dz \; P_{N+1}(x_1, ..., x_N, z, y_1, ..., y_N, z, t) - L\Gamma P_N(\mathcal{C}, t), \tag{5}$$

where the first term on the RHS describes the motion of the steps, and the second term describes the pairwise annihilation of steps at a position z.

a.) We make the ansatz that the stationary distribution $P_N^*(\mathcal{C})$ is constant on the subspaces Ω_N of fixed N, i.e.

$$P_N^*(\mathcal{C}) = \frac{1}{Z} Q_N. \tag{6}$$

Show that this ansatz solves (5) and that $Q_N = \eta^{2N}$ with $\eta = \sqrt{\Gamma/2c}$.

b.) The normalization constant Z in (6) is defined by

$$Z(L, \eta) = \sum_{N=0}^{\infty} Q_N v_N, \tag{7}$$

where $v_N = \int_{\Omega_N} dx_1 ... dx_N dy_1 ... dy_N$ is the volume of the subspace Ω_N. Show that $Z(L, \eta) = I_0(2L\eta)$, where I_0 is a modified Bessel function.
Hint: Steps of equal sign are indistinguishable.

c.) Using the second relation in (3), show that the stationary growth velocity is given by

$$V(u = 0, L) = 2c\eta \frac{I_0'(2\eta L)}{I_0(2\eta L)}. \tag{8}$$

Verify that for $\eta L \gg 1$ this reduces to the result derived above in Exercise 24 a.). Evaluate the expression (8) in the opposite limit $\eta L \ll 1$, and interpret the result: Why is $V \sim L$ in this case?

[2] C.H. Bennett et al., J. Stat. Phys. **24** (1981) 419; N. Goldenfeld, J. Phys. A **17** (1984) 2807.