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Sommersemester 2012

11. Exercise sheet to the lecture "Statistical Physics Far from Equilibrium"

Exercise 26: A mechanism for power-law distributions of extinction times

We use a simple model for the extinction of species[1] to illustrate a mathematical mechanism through which power law distributions can emerge from random variables with an exponential or bounded distribution. Consider an 'ecology' where each species i is characterized by a 'fitness' value $f_i > 0$, which is a random variable drawn from a probability density $p_f(f)$. The 'environment' is characterized by a random process in discrete time t, where a new random variable $u_t \geq 0$ is drawn independently from a probability density $p_u(u)$ in each time step. Species i survives at time step t if $f_i > u_t$, otherwise it goes extinct and is replaced by a new species with randomly drawn fitness. Compute the probability $S(t)$ that a random species survives at least t time steps, for the following two cases:

a.) exponential: $p_f(f) = \alpha e^{-\alpha f}$, $p_u(u) = \beta e^{-\beta u}$.

b.) bounded: $f, u \in [0, 1]$ and $p_f(f) = (\alpha + 1)(1 - f)^\alpha$, $p_u(u) = (\beta + 1)(1 - u)^\beta$ with $\alpha, \beta > -1$.

Under what conditions is the mean lifetime of a species finite/infinite in the two cases?

Exercise 27: Vicsek's snow flake

Generalize the Vicsek snow flake fractal[2] described in the lectures to d embedding dimensions, and compute the fractal dimension D as a function of d. Does this model provide a good description of diffusion-limited aggregation clusters?

Exercise 28: Generalized scale invariance

Sittler and Hinrichsen[3] generalized the notion of scale invariance for functions of several variables to a situation where the scaling exponents $y_1, ..., y_n$ themselves depend on the independent variables $x_1, ..., x_n$. In this setting, the function $F(x_1, ..., x_n)$ of n variables is said to display generalized scale invariance if for any scale factor $b > 0$

$$F(b^{y_1(\vec{x})} x_1, ..., b^{y_n(\vec{x})} x_n) = b^\alpha F(x_1, ..., x_n). \tag{1}$$

Here $\vec{x} = (x_1, ..., x_n)$ and the exponent $\alpha = 1$ without loss of generality.

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[1] See M.E.J. Newman, Proc. Roy. Soc. B **263** (1996) 1605.
[2] T. Vicsek, J. Phys. A: Math. Gen. **16** (1983) L647.
[3] J. Phys. A **35**, 10532 (2002).

a.) Show that (1) implies

$$\vec{y} \cdot \tilde{\nabla}\phi = 1, \tag{2}$$

where $\tilde{\nabla}$ refers to the gradient with respect to the logarithmic variables $X_i = \ln x_i$ and $\phi(X_1, ..., X_n) = \ln F(x_1, ..., x_n)$.

b.) Show that the group structure of rescaling transformations implies the constraint

$$(\vec{y} \cdot \tilde{\nabla})\vec{y} = 0 \tag{3}$$

on the exponent functions $\vec{y} = (y_1, ..., y_n)$.

c.) Show that (3) allows for non-constant solutions of the form

$$y_i = \frac{X_i}{X_n} h_n(X_1/X_n, ..., X_{n-1}/X_n), \tag{4}$$

where h_n is a function of $n-1$ variables, and any other of the n variables X_i could equally well have been chosen for elimination.

d.) Specializing to the case $n = 2$, show that the solution (4) leads to a scaling form

$$\frac{\ln F(x_1, x_2)}{\ln x_2} = G\left(\frac{\ln x_1}{\ln x_2}\right)$$

in the original variables[4].

[4]This type of scaling has been observed, for example, in simulations of one-dimensional sandpile models of self-organized criticality, see Kadanoff et al., Phys. Rev. A **39** (1989) 6524.