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4. Exercise sheet to the lecture "Statistical Physics Far from Equilibrium"

Exercise 7: Entropy production for the master equation

It was shown in the lectures that the entropy production σ for a continuous time Markov chain governed by transition rates Γ_{ij} is given by

$$\sigma = \frac{1}{2}\sum_{i,j}(\Gamma_{ij}P_i - \Gamma_{ji}P_j)\ln\left[\frac{\Gamma_{ij}P_i}{\Gamma_{ji}P_j}\right] = \sigma_{\text{int}} + \sigma_{\text{ext}}$$

where

$$\sigma_{\text{int}} = \frac{1}{2}\sum_{i,j}(\Gamma_{ij}P_i - \Gamma_{ji}P_j)\ln\left(\frac{P_i}{P_j}\right)$$

is the internal entropy production of the system, and σ_{ext} is the entropy flow to the environment. Using the master equation, show that indeed $\sigma_{\text{int}} = \frac{d}{dt}S_{\text{sys}}$, where

$$S_{\text{sys}} = -\sum_i P_i \ln P_i$$

is the (Shannon) entropy of the system. This implies, in particular, that $\sigma_{\text{int}} = 0$ in a stationary state.

Exercise 8: Equal probability of microstates and Bernoulli measure

For the one-dimensional asymmetric exclusion process with N particles on a ring of L sites it has been shown that all $\binom{L}{N}$ microstates are equally likely in the nonequilibrium stationary state. Prove that this implies *Bernoulli measure* in the limit $L, N \to \infty$ at fixed density $\rho = N/L$, which means that in the infinite system each site is independently occupied or vacant with probability ρ and $1-\rho$, respectively. To this end, compute correlation functions $\langle \eta_i \eta_j \rangle$, $\langle \eta_i \eta_j \eta_k \rangle$ etc. for the finite system, and take the limit $L, N \to \infty$.

Exercise 9: Two particles on a ring of four sites

Consider the totally asymmetric exclusion process with two particles on a ring of four sites. In the lectures the transitions among the 6 states of the system were illustrated in a graph. Generalize this diagram to the case of discrete time dynamics (parallel update with probability π). Which new transitions appear? How does the diagram change in the deterministic limit $\pi \to 1$, and what does this imply for the stationary state in this case?

Exercise 10: Cellular automaton rule 184

Cellular automata (CA) are dynamical systems with discrete spatial structure evolving in discrete time. Elementary CA in the sense of Wolfram[1] are defined on a one-dimensional lattice with binary variables on each site, and the state of a site at time t is a deterministic (Boolean) function of the state of the site itself and its two neighbors at time $t-1$. There is a total of $2^8 = 256$ such CA which can be completely classified. Here we consider rule 184 defined by

$$(000) \to 0,\ (001) \to 0,\ (010) \to 0,\ (011) \to 1,\ (100) \to 1,\ (101) \to 1,\ (110) \to 0,\ (111) \to 1$$

The sequence of final states is a binary representation of the number $184 = 0 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 + 1 \times 2^5 + 0 \times 2^6 + 1 \times 2^7$. Rule 184 is the deterministic limit ($\pi \to 1$) of the discrete time asymmetric exclusion process (dTASEP): In one time step all particles with a vacant neighbor site move *simultaneously* to the right.

a.) Identify the rule number of the deterministic dTASEP with all particles moving deterministically to the *left*.

b.) Determine (by inspection or simulation) the attractor of CA 184 on a finite ring, i.e., the set of configurations that govern the dynamics for $t \to \infty$.
Hint: Begin by considering the case of half filling ($N = L/2$) and show that, similar to the ring of four sites examined in Exercise 9, the attractor consists of only two configurations. The case of general N can then be described as a 'gas' of defects moving on the background defined by these two configurations.

c.) Based on the results of part b.), determine the average speed v of particles in the stationary state and the corresponding stationary particle current $J(\rho) = \rho v$, where $\rho = N/L$ is the particle density.

d.) Can you prove that rule 184 and its mirror image considered in part a.) are the only non-trivial rules that conserve the number of 1's?

[1] S. Wolfram, Rev. Mod. Phys. **55**, 601 (1983); S. Wolfram, *A new kind of science* (2002)