UNIVERSITÄT ZU KÖLN

Prof. Dr. Joachim Krug
Dr. Ivan Szendro
Institut für Theoretische Physik
Sommersemester 2012

5. Exercise sheet to the lecture "Statistical Physics Far from Equilibrium"

Exercise 11: Exclusion process with long-ranged jumps

We consider a variant of the totally asymmetric exclusion process on the ring, where particles can perform long-ranged jumps. Specifically, each particle jumps (in continuous time, that is, with exponentially distributed waiting times) as far as it can without overtaking the particle in front of it: If the particle has a cluster of $k \geq 1$ vacant sites in front of it, it jumps a distance k to the right (see Fig.1).

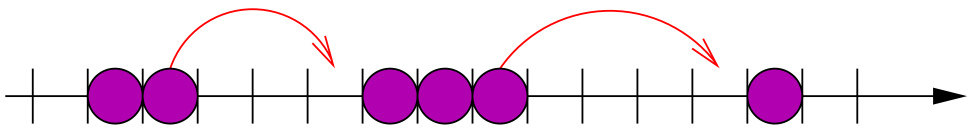

Figure 1: Illustration of the long-ranged exclusion process.

a.) Using the approach explained in the lectures, show that (like for the ASEP) also for this model the distribution that gives equal weight to all $\binom{L}{N}$ configurations is invariant under the dynamics.

b.) As a consequence of part a.), we conclude that the particles are distributed independently on the infinite lattice (Bernoulli measure, see Exercise 8). Use this to derive the stationary current $J(\rho)$ as a function of the density. To this end, first determine the probability that a particle has k vacant sites in front of it, and then sum over k.

c.) Using the result of part b.), you can now write down the hydrodynamic equation. How does an arbitrary initial density profile $\rho(x, 0)$ evolve? Can shocks form in this model?

d.) To check the above results, perform a computer simulation of the system. What do you see happening? What is wrong with the arguments in parts a.) and b.) ?

Exercise 12: Shock formation in CA 184

In Exercise 10 it was shown that the current-density relation of the CA 184 is

$$J(\rho) = \min(\rho, 1 - \rho).$$

Consider the hydrodynamic equation associated with this current. Examine the behavior of the characteristics. Under what conditions and on which time scale do shocks form from a smooth initial density profile?

Exercise 13: Shocks in the viscous Burgers equation

The one-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + \lambda u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{1}$$

was originally introduced as a model problem for fluid turbulence[1]. In that context $\lambda = 1$ because of Galilean invariance, and $\nu > 0$ is the kinematic viscosity.

The inviscid Burgers equation with $\nu = 0$ describes the behavior of the asymmetric exclusion process in the hydrodynamic limit. It was shown in the lectures that the inviscid equation generically develops shock discontinuities from smooth initial conditions. Here we want to investigate how the viscosity regularizes the shocks.

Show that (1) admits stationary traveling wave solutions of the form

$$u(x, t) = U(x - Vt), \tag{2}$$

where the shape function is asymptotically constant, $\lim_{\xi \to -\infty} U(\xi) = u_L$ and $\lim_{\xi \to \infty} U(\xi) = u_R$. Find a first integral of the ordinary differential equation for $U(\xi)$ and use it to determine the velocity V in terms of the boundary values u_L and u_R. Then compute the function $U(\xi)$ explicitly and show that the solution reduces to a discontinuous shock in the limit $\nu \to 0$.

[1] J.M. Burgers, *The Nonlinear Diffusion Equation* (Riedel, Boston 1974).